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Some Mixed Quadrature Rules for Approximate Evaluation of Real Cauchy Principal Value Integrals

[1]R.N.Das, G.Pradhan[2], Swagatika Das[3]
[1]*Ex. Professor of Mathematics and Computer science School of Mathematics and Computing Gandhi Institute for Technological Advancement Badaraghunathpur, Jaanla, Pin: 752054 Bhubaneswar, India*
[2]*Head of the department of mathematics & Humanities College of Engineering & Technology Bhubaneswar-751003, India*
[3]*Department of mathematics & Humanities College of Engineering & Technology. Bhubaneswar-751003, India*

Abstract:- In this paper some mixed quadrature rules have been constructed for numerical integration of real Cauchy principal value integrals and their asymptotic error estimates have been derived. The numerical verification of the rules has been done by considering some standard Cauchy principal value integrals.

Keywords:- CPV integral; basic rules; mixed quadrature rule; asymptotic error estimate, numerical integration.
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I. INTRODUCTION

Das and Pradhan [7] have introduced the concept of mixed quadrature rule for numerical integration of real definite integrals. In this technique they have combined in a suitable way two quadrature rules: one is a Newton cotes type rule and the other is a Gauss type rule; the rules are of equal precision. The combined rule or the so called mixed quadrature rule is of precision higher than each of the rules considered in combination. In fact, the degree of precision of mixed quadrature rule is d+2; if each of the rules in combination is of precision d.

In this paper we have made an attempt to combine two rules of equal precision meant for the numerical evaluation of Cauchy principal value integral to form a mixed quadrature rule of higher precision for determination of approximate value of a real CPV integral of the type

$$I(f) = \int_{-a}^{a} \frac{f(x)}{x} dx; \ \ a > 0 \tag{1}$$

The formulation of the rule is outlined in the next section.

II. CONSTRUCTION OF THE RULE OF PRECISION FOUR

First of all, we construct a rule having the the nodes:

$$\frac{-a}{\sqrt{3}} ,0 ,\frac{a}{\sqrt{3}}$$

for the approximate evaluation of the real Cauchy Principal Value integrals of the type given in equation (1). This rule is of precision four which will be combined with another rule of precision four to yield a precision of six for the approximation of integral (1).

Let the rule based on these three points be denoted by $R_1(f)$ and suppose

$$R_1(f) = W_0 f(0) + W_1 \left[f\left(\frac{a}{\sqrt{3}}\right) - f\left(\frac{-a}{\sqrt{3}}\right)\right] \tag{2}$$

where W_1 and W_2 are the weights of the rule and are to be determined so that ;

$$I(x^k) = R_1(x^k) \quad \text{for k = 0,1,3.} \tag{3}$$

Further it may be noted here that

$$I(x^k) = R_1(x^k) \quad \text{for k = 2 and for even k} \tag{4}$$

since the proposed rule given in (2) meant for approximate evaluation of (1) is a symmetric quadrature rule.

Using the identities given in equation (3), the following values for the weights W_0, W_1 are obtained

$$W_0 = 0, W_1 = \sqrt{3} \ . \tag{5}$$

So the rule given in equation (2) becomes

$$R_1(f) = \sqrt{3} \left[f\left(\frac{a}{\sqrt{3}}\right) - f\left(\frac{-a}{\sqrt{3}}\right)\right] \tag{6}$$

which is practically a two point rule, since the coefficient of f(0) is found to be zero.

Let E_1(f) denote the truncation error in approximating the integral I(f) given in equation (1) by the rule (6).

Then $$I(f) = R_1(f) + E_1(f). \tag{7}$$

It is easy to note that

$$E_1(x^k) = \begin{cases} 0; \; k = 0(1)4 \\ \frac{8}{45} \; ; k = 5 \end{cases} \quad (8)$$

which implies that the quadrature rule given in equation (6) exactly integrates all monomials of degree less than or equal to four.
So, the degree of precision of the rule given in (6) is four.

III. CONSTRUCTION OF MIXED RULES

To construct the mixed rule of precision six we consider here, the following two rules each of precision four:

$$\int_{-a}^{a} \frac{f(x)}{x} dx \approx R_1(f) = \sqrt{3}\left[f\left(\frac{a}{\sqrt{3}}\right) - f\left(\frac{-a}{\sqrt{3}}\right)\right] \quad (9)$$

given in equation (6) and the rule

$$\int_{-a}^{a} \frac{f(x)}{x} dx \approx R_2(f) = \frac{16}{9}\left[f\left(\frac{a}{2}\right) - f\left(\frac{-a}{2}\right)\right] + \frac{1}{9}[f(a) - f(-a)] \quad (10)$$

due to Das and Hota [6] .

Let us assume that $E_2(f)$ is the truncation error associated with the rule $R_2(f)$ given in equation (10) i.e.

$$I(f) = R_2(f) + E_2(f). \quad (11)$$

Now suppose that f(x) is sufficiently differentiable in the interval $-1 \le x \le 1$ and using the Taylor's expansion of f(x) about the point '0' the errors E_1 (f) and E_2 (f) associated with the rules (9) and (10) respectively are obtained as

$$E_1(f) = \frac{8}{45} a^5 \frac{f^{(5)}(0)}{5!} + \frac{40}{189} a^7 \frac{f^{(7)}(0)}{7!} + \ldots \quad (12)$$

and

$$E_2(f) = \frac{1}{15} a^5 \frac{f^{(5)}(0)}{5!} + \frac{1}{28} a^7 \frac{f^{(7)}(0)}{7!} + \ldots \quad . \quad (13)$$

So,

$$I(f) = R_1(f) + \frac{8}{45} a^5 \frac{f^{(5)}(0)}{5!} + \frac{40}{189} a^7 \frac{f^{(7)}(0)}{7!} + \ldots \quad (14)$$

and

$$I(f) = R_2(f) + \frac{1}{15} a^5 \frac{f^{(5)}(0)}{5!} + \frac{1}{28} a^7 \frac{f^{(7)}(0)}{7!} + \ldots \quad . \quad (15)$$

Now multiplying (15) by $(-\frac{8}{3})$ and then adding with equation (14) we obtain after simplification,

$$I(f) = \frac{1}{5}(8R_2 - 3R_1) + \frac{1}{5}(8E_2 - 3E_1)$$
$$= R_{1,2}(f) + E_{1,2}(f); \text{ say .}$$

where $\quad R_{1,2}(f) = \frac{1}{5}(8R_2 - 3R_1) \quad (16)$

and $\quad E_{1,2}(f) = \frac{1}{5}(8E_2 - 3E_1)$

The desired mixed quadrature rule is $R_{1,2}(f)$ and the associated truncation error in approximation is $E_{1,2}(f)$, which is given as

$$E_{1,2}(f) = -\frac{22}{315} a^7 \frac{f^{(7)}(0)}{7!} - \frac{7}{54} a^9 \frac{f^{(9)}(0)}{9!} - \ldots \quad . \quad (17)$$

Now

$$E_{1,2}(x^k) = \frac{1}{5}[8E_2(x^k) - 3E_1(x^k)]$$
$$= 0; \quad k = 0(1)4.$$

and $\quad E_{1,2}(x^5) = 0$

which implies that the degree of precision of the mixed rule $R_{1,2}(f)$ is at least five . Further, as the rule is a combination of two symmetric quadrature rules, it follows that

$$E_{1,2}(x^k) = 0 \; ; \; for \; k=6.$$

However, $\quad E_{1,2}(x^7) = -\frac{22}{315} \ne 0$

which finally proves that the rule $R_{1,2}(f)$ has degree of precision six.

Following the procedure for the construction of the mixed quadrature rule $R_{1,2}(f)$ of precision six given in equation (16) from two rules, each of precision four given in equation (9) and (10) respectively, we have obtained the following mixed quadrature rule which we denote by $R_{1,2,3}(f)$ and is given by

$$R_{1,2;3}(f) = \frac{1}{161}\left(297R_3(f) - 136R_{1,2}(f)\right) \quad (18)$$

from the rule $R_{1,2}(f)$ given in equation (16) and the rule

$$R_3(f) = \frac{33}{20}\left[f\left(\frac{a}{3}\right) - f\left(-\frac{a}{3}\right)\right] + \frac{12}{25}\left[f\left(\frac{2a}{3}\right) - f\left(-\frac{2a}{3}\right)\right] + \frac{13}{100}[f(a) - f(-a)] \quad (19)$$

due to Das and Hota [6]. The rule $R_{1,2,3}(f)$ can be easily shown to be of precision eight.

The truncation error associated with the rule $R_{1,2,3}(f)$ which is denoted by $E_{1,2,3}(f)$ is given by

$$E_{1,2;3}(f) = \frac{1}{161}\left(297E_3(f) - 136E_{1,2}(f)\right)$$
$$= -\frac{428}{65205}a^9\frac{f^{(9)}(0)}{9!} - \frac{73}{7128}a^{11}\frac{f^{(11)}(0)}{11!} + \dots \quad . \tag{20}$$

If at this stage, the above described procedure is continued one more step with the rule $R_{1,2;3}(f)$ given in equation (18) and the rule

$$R_4(f) = \frac{9152}{4725}\left[f\left(\frac{a}{4}\right) - f\left(-\frac{a}{4}\right)\right] + \frac{592}{4725}\left[f\left(\frac{a}{2}\right) - f\left(-\frac{a}{2}\right)\right] + \frac{1856}{3675}\left[f\left(\frac{3a}{4}\right) - f\left(-\frac{3a}{4}\right)\right] + \frac{2459}{33075}\left[f(a) - f(-a)\right] \tag{21}$$

due to Price [8] (which is also obtained as a special case of Newton-Cotes type of quadrature rule for approximation of real CPV integral due to Das and Hota [6]), we obtain the following rule

$$R_{1,2;3,4}(f) = \frac{109568}{62579}R_4(f) - \frac{46989}{62579}R_{1,2,3}(f) \tag{22}$$

which can be shown to be of precision ten.

The corresponding truncation error is given by

$$E_{1,2;3,4}(f) = \frac{1}{62579}\left(109568\,E_4(f) - 46989E_{1,2,3}(f)\right)$$
$$= -\frac{3198665}{346937976}a^{11}\frac{f^{(11)}(0)}{11!} + \dots \quad . \tag{23}$$

This procedure for construction of mixed quadrature rule of precision (d+2), from two rules of same precision'd' can be continued to obtain rules of higher precision d (>10).

IV. ERROR ANALYSIS

The asymptotic error estimates of all the mixed quadrature rules constructed in the preceding article are given in the following theorem.

Theorem:

Let f(x) be sufficiently differentiable in the interval [-1,1]. Then the errors $E_{1,2}$(f), $E_{1,2,3}$(f) and $E_{1,2,3,4}$(f) associated with the rules $R_{1,2}$(f), $R_{1,2,3}$(f) and $R_{1,2;3,4}$(f) respectively are given by

$$\left|E_{1,2}(f)\right| \approx \frac{22}{315}\frac{a^{(7)}}{7!}\left|f^{(7)}(0)\right| \tag{24}$$

$$\left|E_{1,2,3}(f)\right| \approx \frac{428}{65205}\frac{a^{(9)}}{9!}\left|f^{(9)}(0)\right| \tag{25}$$

and

$$\left|E_{1,2;3,4}(f)\right| \approx \frac{3198665}{346937976}\frac{a^{(11)}(0)}{11!}\left|f^{(11)}(0)\right| \quad . \tag{26}$$

Proof: The asymptotic error estimates of the truncation errors given in equations (24), (25) and (26) follow from equations (17), (20) and (23) respectively.

V. NUMERICAL VERIVICATION

The approximate values of the integrals

$$I_1 = \int_{-1}^{1}\frac{e^x}{x}\,dx$$
$$I_2 = \int_{-1}^{1}\frac{sinx}{x}\,dx$$

and

$$I_3 = \int_{-1}^{1}\frac{e^x(x+1)}{x}\,dx$$

have been numerically evaluated by using the rules $R_{1,2}(f)$, $R_{1,2,3}(f)$ and $R_{1,2,3,4}(f)$ of degree of precision six, eight and ten respectively and the approximate values along with their exact values are given in Table-2.

The rules:$R_k(f)$; k =1,2,3,4 considered in this paper for construction of mixed quadrature rules are called basic rules with reference to the mixed quadrature rules and the approximate values of the integrals I_k; k= 1,2,3 obtained by using these basic rules are depicted in Table:1 which have reused to evaluate approximately the values of the same integrals by mixed quadratures rules using the formulae given in equations: (16), (18) and (22).

All computations has been performed by using c^{++}.

Table I

Integrals→ Rules↓	**Approximate value of I_1**	**Approximate value of I_2**	**Approximate value of I_3**
R_1	2.11297772	1.890726111	4.45567381

R_2	2.11393913	1.891617689	4.46151432
R_3	2.11450827	1.892159966	4.46495666
R_4	2.11450176	1.892166148	4.46490422
Exact value →	2.11450175	1.892166141	4.46490413

Table II

Integrals→ Rules↓	**Approximate value of I_1**	**Approximate value of I_2**	**Approximate value of I_3**
$R_{1,2}(f)$	2.11451597	1.892152635	4.46501862
$R_{1,2,3}(f)$	2.11450177	1.892166159	4.46490433
$R_{1,2,3,4}(f)$	2.11450175	1.892166141	4.46490413
Exact value →	2.11450175	1.892166141	4.46490413

VI. CONCLUSION

When the sample integrals are approximated by the mixed rules obtained, it is observed that the approximate values in succeeding steps approach to the exact values and agree to almost eight decimal places of accuracy in the present case. From the observed trend of approximations, the approximate values accurate to certain desired decimal figure can also be achieved if approximation is continued by using mixed rules of precision more than ten.

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