Bounds on Metric Dimension

*K.Renganathan[1], M.Arockia RanjithKumar[2]

[1]*SSM College of Engineering and Technology, Department of Mathematics, Dindigul, Tamil nadu, India.*
[2]*DMI-St.Eugene University, Department of Science and Humanities, Chibombo, Zambia.*
Corresponding Author: K.Renganathan

ABSTRACT: In this paper, we present some bounds for metric dimension of a graph G in terms of order and some theoretic parameters such as diameter and maximum degree etc., Also, we characterize the Extremal graphs achieving the bounds.
Keywords: *Metric bounds, Distance partition, Metric dimension, Extremal graph.*

I. INTRODUCTION

For any graph theoretic parameter, the study of determining bounds is the important one. Chartrand et. al [4] determined the bounds of the metric dimensions for any connected graphs and determine the metric dimension of some well-known families of graphs such as paths and complete graphs. In [10], Khuller et. al considered graphs with small metric dimension and showed that a graph has metric dimension 1 if and only if it is a path and Chartrand et. al also proved this in [6]. Buczkowski et. al [1] proved the existence of a graph G with $\beta(G)=2$, for every integer $k\geq2$. In this paper, we present some bounds for metric dimension of a graph G in terms of order and some theoretic parameters such as diameter and maximum degree etc.,

1.1 Some Bounds For Metric Dimension
1.2 Some Bounds for Metric dimension

In this section, we determine some bounds for metric dimension and characterize the graph with metric dimension l and n-1. Also we characterize the Extremal graphs achieving the bounds.

Theorem 1.1.1.

If G is a graph on n vertices, then $1 \leq \beta(G) \leq n-1$. For given integer a and n with $1 \leq a \leq n-1$, there exists a graph G of order n such that $\beta(G)=a$.
Proof: The inequalities are trivial. Now suppose a and n are two integers with $1 \leq a \leq n-1$. We construct a graph G of order n such that $\beta(G)=a$ as follows.

Case 1. a=1, 2, n-1, n-2

For a=1, 2, n-1, n-2, let G be a graph with n-vertices be taken as a path, cycle, complete graph and complete bipartite graph respectively. Then clearly $\beta(G)=a$.
Case 2. $3 \leq a \leq n-3$ and n-a is odd.

In this case, let G be a graph obtained from the cycle $3 \leq a \leq n-3C_{n-a+1}=(v_1, v_2, \ldots, v_{n-a+1}, v_1)$ by attaching (a-1) pendent edges at any one of the vertices of the cycle say v_1 and let $x_1, x_2, \ldots, x_{a-1}$ be the pendent vertices of G. We now claim that $\beta(G)=a$.

Let $S=(x_1, x_2, \ldots, x_{a-2}, v_2, v_{n-a+1})$. It can be easily verified that S is a resolving set of G. So that $\beta(G) \leq |S = a|$. Next we have to show that $\beta(G) \geq a$. For that, we have to prove the following Claim 1.

Claim 1. Every resolving set of G contains at least a-2 vertices from the set $X=\{x_1, x_2, \ldots, x_{a-1}\}$.

Suppose not, then there exists a resolving set of G contains at most a-3 vertices say W_1 and so $|X - W_1| \geq 2$, However if $x_i, x_j \in X-W_1$, then $d(x_i, v)=d(x_j, v)$, $\square$ $v \in V(G)$. Hence no vertex of W_1 resolves x_i and x_j, a contradiction. This complete the proof of Claim 1.

Use the fact $\beta(C_n)=2$ and Claim 1 we have $\beta(G) \geq a-2+2$ and hence $\beta(G)=a$.

Case 3: $3 \leq a \leq n-3$ and n-a is even.

Here also let G be the graph obtained from the cycle $C_{n-a+1}=(v_1, v_2, \ldots, v_{n-a+1}, v_1)$ by attaching (a-1) pendent edges at any one of the vertices of the cycle say v_1 and attach one pendent edge at any other vertices of the cycle say v_1. Let $x_1, x_2, \ldots, x_{a-1}$ be the pendent vertices of G, where x_a is incident with the pendent edge which is attached to v_2. We now claim that $\beta(G)=a$.

Let $S=(x_1, x_2, \ldots, x_{a-2}, x_a, v_{n-a})$. Then it can be easily verified that S is a resolving set of G and so $\beta(G) \leq |S| =a$. Next we have to prove that $\beta(G)\geq a$. For that, first we prove the following Claim 2.

Claim 2. Every resolving set of G contains at least 2 vertices from the set $T=C_{n-a} \cup \{x_a\}$.

Assume to the contrary, then there exists a resolving set of G contains at most one vertex from T say W_2. Note that if v_i and v_j are two distinct vertices of C_{n-a} with $d(v_i, v_1)=d(v_j, v_1)$ then $d(v_i, v)=d(v_j, v)$ for all $v \in V(G)-C_{n-a} \cup \{x_a\}$, it follows that W_2 must contain exactly one vertex in T. We consider the following four cases.

Case (i). $x_a \in W_2$.

Since for any $v' \in V(G)-C_{n-a} \cup \{x_a\}$, $d(v_{n-a}, x_a)=d(v', x_a)$ and $d(v_{n-a}, u')=d(v', u')$ for any $u' \in V(G)-C_{n-a} \cup \{x_a\} \cup \{v'\}$ it follows that $r(v_{n-a}\backslash W_2)=r(v'\backslash W_2)$.

Case (ii). Any one of $\{v_1, v_2, \ldots, v_{n/2-1}\}$ belongs to W_2.

Since for any $v' \in V(G)-C_{n-a} \cup \{x_a\} \cup \{v'\}$, $d(v_{n-a}, v_i)=d(v', v_i)$, $1\leq i \leq n/2-1$ and $d(v_{n-a}, u')=d(v', u')$ for all $u' \in V(G)-C_{n-a} \cup \{x_a\} \cup \{v_i\}$, we have $r(v_{n-a}\backslash W_2)=r(v'\backslash W_2)$.

Case (iii). $V_{n/2} \in W_2$.

Note that if v and v' are two distinct vertices of C_{n-a} with $d(v, v_1)=d(v',v_1)$ then $d(v, v_{n/2})$ and $d(v, u)=d(v', u)$ for all $u \in V(G)-C_{n-a}\{x_a\}$ and so $r(v_2\backslash W_2)=r(v'\backslash W_2)$.

Case (iv). Any one of $\{v_{n/2}+1, \ldots, v_{n-a}\}$ belongs to W_2.

Since for any $v' \in V(G)-C_{n-a} \cup \{x_a\}$, $d(v_2, v_i)=d(v', v_i)$, $n/2+1\leq i \leq n-a$ and $d(v_2, u')=d(v', u')$ for all $u' \in V(G)-C_{n-a} \cup \{x_a\} \cup \{v_i\}$, we have $r(v_2\backslash W_2)=r(v'\backslash W_2)$.

In each case, W_2 is not a resolving set of G, a contradiction. Therefore, every resolving set of G contains at least two vertices from the set T. From Claim 1. and Claim 2. $\beta(G)\geq a$ and hence $\beta(G)=a$.

Illustration (i). If n=10 and a=5, then the required graph G is given Figure 1.1.1. This is actually discussed in Case 2. One can verify that $\beta(G)=5$.

Illustration (ii). If n=12 and a=4, then the required graph G is given in Figure 1.1.2. This is actually discussed in Case 3. One can easily verify that $\beta(G)=4$.

In the Theorems 1.1.2. and 1.1.3., we characterize the Extremal graphs achieving the bounds given in Theorem 1.1.1.

A. Theorem 1.1.2. A connected graph G of order n has metric dimension 1 if and only if $G \cong P_n$.

Proof: Let G be a graph with $\beta(G)=1$. We have to prove that G is a path.

Let W={w} be a minimum resolving set for G. For each vertex v $\in$ V(G), r(v/W) = d(v,w) is a non negative integer less than n. Since the codes of the vertices of G with respect to W are distinct, there exists a vertex u of G such that d(u,w)=n-1. Consequently, the G be a path on n-vertices. By Proposition 1.1.1. $\beta(G)=1$.

B. Theorem 1.1.3. Let G is a connected graph of order n≥2. Then $\beta(G)=n-1$ if and only if $G \cong K_n$.

Proof: Let G be a graph with $\beta(G)=n-1$. We will show that $G \cong K_n$. Suppose not. Then G contains two vertices u and v with d(u, v)=2. Let u, x, v be a path of length 2 in G and let W=V(G)-[x, v].

Since d(u, v)=2 and d(u, x)=1, it follows that r(x\W)=r(v\W) and so W is a resolving set. Which is contradiction to the fact that $\beta(G)=n-1$. For the converse, assume that $G \cong K_n$. By Proposition 1.1.12. $\beta(G)=n-1$.

In the following theorem we determine some bounds for the metric dimension of a graph in terms of maximum degree and diameter.

C. Theorem 1.1.4. Let G be a nontrivial connected graph of order n≥2, diameter d(G), and maximum degree $\Delta(G)$. Then $[\log_3(\Delta(G)+1)] \leq \beta(G) \leq n-d(G)$.

Proof: First, we establish the upper bound. Let u and v be vertices of G for which d(u ,v)=d(G) and let $u=v_0, v_1, v_2, \ldots, v_{d(G)}=v$ be a shortest u-v path.

Let W=V(G)-$\{v_1, v_2, \ldots, v\}$. Since u$\in$W and d(u, v_i)=I for 1$\leq$ i $\leq$d(G), it follows that W is a resolving set of cardinality n-d(G) for G. Thus $\beta(G) \leq n-d(G)$.

Next, we consider the lower bound. Let $\beta(G)=k$ and let v$\in$V(G) with deg v= Δ. Moreover, let N(v) be the neighbourhood of v and let W=$\{w_1, w_2, \ldots, w_k\}$ be a resolving set of G. Observe that if u$\in$N(v), then for

each $1 \leq i \leq k$, the distance d(u, w$_i$) is one of the numbers d(v, w$_i$), d(v, w$_i$)+1 or d(v, w$_i$)-1. Moreover, since W is a resolving set, r(u\W)=r(v\W) for all u∈N(v). Thus there are three possible number for each of the k coordinates of r(u\W). On the other hand, since it cannot occur that d(u, w$_i$)=d(v, w$_i$) for all i ($1 \leq i \leq k$), it follows that there at most 3^k-1 distinct codes of the vertices in N(v) with respect to W. Therefore, |N(v)|= $\Delta \leq 3^k$ -1, which implies that β(G) = k $\geq$ log$_3$ (Δ(G)+1). Since β(G) is an integer, β(G) $\geq$ log$_3$ (Δ(G)+1).

1.3 Graphs with β = n-2

This section completely characterizes the family of graphs of order n for which the metric dimension n-2.

D. *Theorem 1.2.1.* *Let G be a connected graph of order n $\geq$ 4. Then β = n-2 if and only if G = $K_{s,t}$ (s, t $\geq$1),* $G = K_s + \overline{K}_t$ *,(s $\geq$1, t $\geq$2), or G = K_s+ ($K_1 \cup K_t$)(s,t $\geq$1).*

Proof: It can be easily show that $\beta(G) \leq n-2$ for each of the graphs mentioned in the statement of the theorem. To see $\beta(G) \leq n-2$, note that if the vertices of a graph are partitioned as $V_1 \cup V_2 \cup ... \cup V_p$ where either V_i is independent and its vertices have identical open neighbourhoods, then the metric dimension is at least $(|V_1|-1)+(|V_2|-1)+...+(|V_p|-1)$. Since each of the graphs mentioned in the statement of the theorem are partition as $V_1 \cup V_2$, then the metric dimension is at least $(|V_1|-1)+(|V_2|-1)$. Therefore $\beta(G) \leq n-2$ and hence $\beta(G) \leq n-2$.

For the converse, assume that G is a connected graph of order $n \geq 4$ such that $\beta(G) \leq n-2$. By Theorem 1.1.4. and, it follows that G has diameter 2. If G is bipartite and since the diameter of G is 2, $G = K_{s,t}$ for some integers $s, t > 1$. Hence, we may assume that G is not bipartite. Therefore, G contains an odd cycle. Let C_r be a smallest odd cycle in G. We claim that $r = 3$. Certainly, C_r is an induced cycle of G. If G contains an induced & cycle $v_1, v_2, ..., v_k$ where $k > 5$, then $W = V(G) - \{v_1 > 2 > v_j, v_4\}$ is a resolving set of cardinality $n-3$, for if we let $w_i = v_j$ and $w_2 = v_5$, then $r(v_2 \setminus W) = (1, s, ...), r(v_3 \setminus W) = (2, 2, ...)$ and $r(v_4 \setminus W) = (t, 1, ...)$, where $s, t > 2$. Hence, $p(G) < n-3$, which is a contradiction. Thus G has no induced cycle of length $k > 5$ and so $r = 3$ and G contains a triangle.

Let Y be the vertex set of a maximum clique of G. Since G contains a triangle, |Y|>3. Let $U \approx V(G) - Y$. Since G is not complete, |U| > 1. If |U| > 1, then $G = K_s + K_j$, for some integers s and t. Now, s > 1 since G is connected and t > 1 since G is not complete. From these observations, we may assume that |U| > 2.

First, we show that t is an independent set of vertices. Suppose, to the contrary, that U is not independent. Then U contains two adjacent vertices u and w. Because of the defining property of Y, there exists $v e Y$ such that $uv t E(G)$ and $v'eY$ such that $wv' g E(G)$, where v and v' are not necessarily distinct. We consider the following two cases.

Case 1. There exists a vertex $v e Y$ such that $uv, wv g E(G)$.

The following two cases are to be discussed.

Subcase 1.1. There exists a vertex $x e 7$ that is adjacent to exactly one of u and w, say u.

Figure 1.2.1.

(a)   (b)
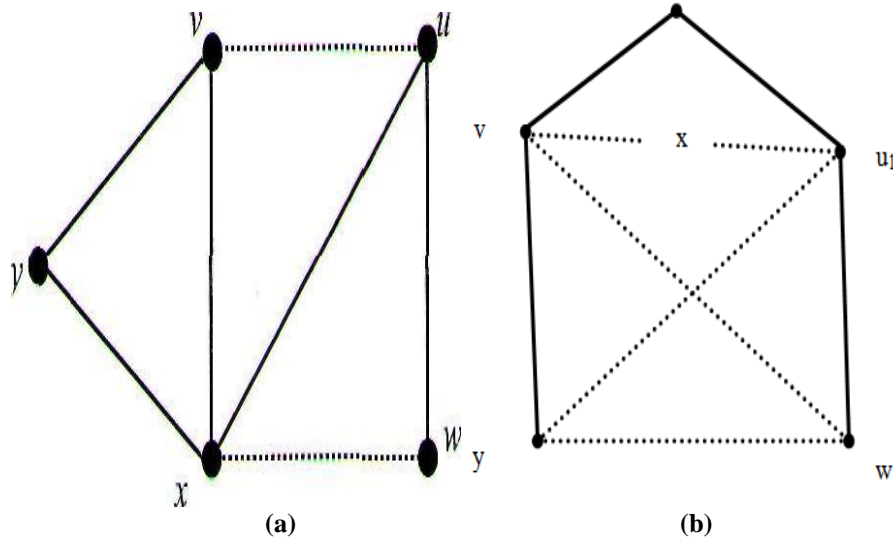
Since |Y| >3, there exists a vertex $y \, e \, Y$ that is distinct from v and x. Thus G contains the subgraph shown in Figure 1.2.1 (a), where dotted lines indicate that the given edge is not present.

Let $W = V(G) - \{u, w, y_j\}$. Letting $w_1 = v$ and $w_2 = x_t$, we have

$$r(u \setminus W) = (2,1,$$
$$r(w \setminus W) = (2,2,...),$$
$$r(y \setminus W) = (1,1,...).$$ So W is a resolving set of cardinality $n - 3$, which is a contradiction.

Subcase 1.2. Every vertex of *F* is adjacent to either both *u* and *w* or to neither *u nor w*.

If *u* and *w* are adjacent to every vertex in $Y - \{v\}$, then the vertices of $(F - \{v\}) \cup \{u, w\}$ are pair wise adjacent, contradicting the defining property of *Y*. Thus, there exists a vertex $v \, e \, Y$ such that *y* is distinct from *v*, and *y* is adjacent to neither *u nor w*.

Since the diameter of *G* is 2, there is a vertex *x* of *G* that is adjacent to both *u* and *v*. Thus *G* contains the subgraph shown in Figure 3.2.1 (b), where dotted lines indicate that the given edges are not in *G*.

Let $W = V(G) - \{x, y, w\}$ and label $w_t - v$ and $w_2 - u$. Then
$$r(x \setminus W) = (1,1,...),$$
$$r(y \setminus W) = (1,2,...),$$
$$r(w \setminus W) = (2,1,...).$$

Thus *W* is a resolving set of cardinality $n - 3$, producing a contradiction.

Case 2. For each vertex v of *Y*, v is adjacent to at least one of u and w.

Because *Y* is the vertex set of a maximum clique, there exist vertices v, $v' e \, Y$ such that $uv, wv' e \, E(G)$. Necessarily, $vw, v'u \, e \, E(G)$. Since $|Y| > 3$, there exists a vertex *y* in *Y* distinct from v and v'. Now, at least one of the edges *yu* and *yw* must be present in *G*, say *yu*. Thus, *G* contains the subgraph shown in Figure 3.2.2

Sample IJERD Paper for A4 Page Size
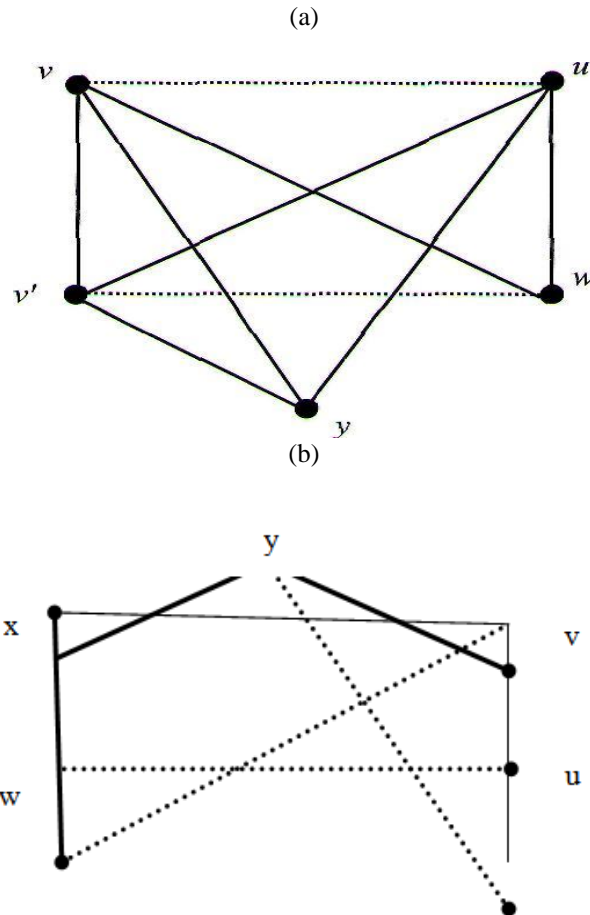
(a) Where again dotted edges indicate that the given edge is not in G.

Let $W = V(G) - \{u, w, y\}$ and label $w_j = v$ and $w_z = v'$. Then
$$r(u \setminus W) = (2, 1, \ldots),$$
$$r(w \setminus W) = (1, 2, \ldots),$$
$$r(y \setminus W) = (1, 1, \ldots).$$

Again, W is a resolving set of cardinality $n - 3$, producing a contradiction.

Thus, as claimed, U is independent.

Figure. 1.2.2.

(a)

(b)

Next, we claim that $N(u) = N(w)$ for all $u, w \, e \, U$. Let u and w be two vertices of U. Suppose that $uv \, e \, E(G)$ for some vertex v of G. Necessarily, $v \, e \, Y$. We show that $wv \, e \, E(G)$. Assume, to the contrary, that $wv \in E(G)$. Since G is connected and U is independent, w is adjacent to some vertex of Y. If w is adjacent only to y, then since w and y are not adjacent to u, $d(w, u) = 3$, which contradicts the fact that the diameter of G is 2. Thus there exists a vertex x in 7 distinct from y such that $wx \, e \, E(G)$. Therefore, G contains the subgroup shown in Figure 3.2.2 (b), where again dotted edges are not in G.

Let $W = V(G) - \{u, w, x\}$ and label $w_j = v$ and $w_z = y$. Then
$$r(u \setminus W) = (1, 2, \ldots),$$
$$r(w \setminus W) = (2, \ldots),$$

41

$r(x \setminus W) = (1,1,...)$.

Thus, W is a resolving set of cardinality $n-3$, producing a contradiction.
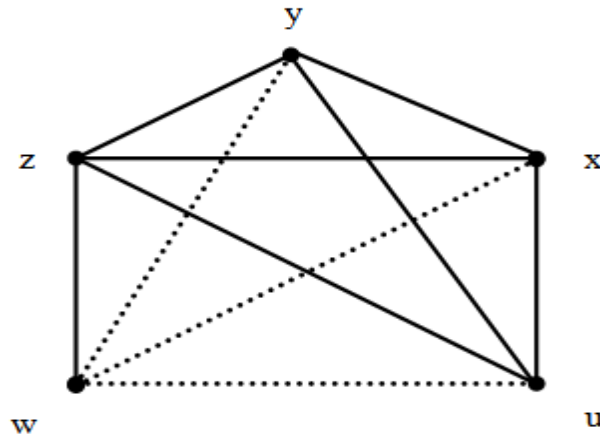

Figure. 1.2.3

Therefore $V(G) = Y \cup U$, where Y induces a clique, U is independent, $|Y| \geq 3, |U| \geq 2$ and $N(u) = N(w)$ for all $u, w \in U$.

Next, we claim that for $u \in U$, there is at most one vertex of Y not contained in $N(u)$. Suppose, to the contrary, that there are two vertices $x, y \in Y$ not in $N(u)$. Let W be a vertex of U that is distinct from u. Therefore, $N(u) = N(w)$. Since G is connected, there exists $z \in 7$ such that: $z \in N(u) = N(w)$. Thus G contains the subgraph shown in Figure 1.2.3., where dotted edges are not edges of G.

Let $W = V(G) - \{y, w, z\}$ and label $w_j = x$ and $w_z = u$. Then
$r(y \setminus W) = (1, 2, ...)$,
$r(w \setminus W) = (2, 2, ...)$,
$r(z \setminus W) = (1, 1, ...)$.

Hence, W is a resolving set of cardinality $n-3$, producing a contradiction.

Now, $N(u) = Y$ For $N(u) = Y - \{v\}$ for some $v \in 7$. If $N(u) = Y$, then $G = K_s + K_t$ for $s = |Y| \geq 3$ and $t = |U| \geq 2$. If $N(u) = Y - \{v\}$, then $G = K_s + (K_1 \cup K_{\bar{t}})$, where $V(K_1) = \{v\}$, $S = |Y| - 1 \geq 2$ and $T = |U| \geq 2$.

However, $K_s + (K_1 \cup K_{\bar{t}}) = K_s + K_{\overline{t+1}}$. In either case, G is the join of a complete graph and an empty graph.

II. CONCLUSIONS

Navigation can be studied in a graph-structured frame work in which the navigating agent (which we shall assume to be a point robot) moves from node to node of a "graph space". The robot can locate itself by the presence of distinctively labeled "landmark" nodes in the graph space. For a robot navigating in Euclidean space, visual detection of a distinctive landmark provides information about the direction to the landmark, and allows the robot to determine its position by tr   'ation. On a graph, however, there is neither the concept of direction nor that of visibility. Instead, v x ll assume that a robot navigating on a graph can sense the distances to a set of landmarks. Evident   the robot knows its distances to a sufficiently large set of landmarks, its position on the graph is uniquely determined. This suggests the following problem:"Given a graph, what are the fewest number of landmarks needed, and where should they be located, so that the distances

to the landmarks uniquely determine the robot's position on the graph?'This is actually a classical problem about metric spaces. A minimum set of landmarks which uniquely determine the robot's position is called a "metric basis", and the minimum number of landmarks is called the "metric dimension" of the graphThis dissertation deals with this notion of metric basis and the corresponding parameter metric dimension.

REFERENCES

[1]. S. M. Metev and V. P. Veiko, Laser Assisted Microtechnology, 2nd ed., R. M. Osgood, Jr., Ed. Berlin, Germany: Springer-Verlag, 1998.

[2]. J. Breckling, Ed., The Analysis of Directional Time Series: Applications to Wind Speed and Direction, ser. Lecture Notes in Statistics. Berlin, Germany: Springer, 1989, vol. 61.

[3]. S. Zhang, C. Zhu, J. K. O. Sin, and P. K. T. Mok, "A novel ultrathin elevated channel low-temperature poly-Si TFT," IEEE Electron Device Lett., vol. 20, pp. 569–571, Nov. 1999.

[4]. M. Wegmuller, J. P. von der Weid, P. Oberson, and N. Gisin, "High resolution fiber distributed measurements with coherent OFDR," in Proc. ECOC'00, 2000, paper 11.3.4, p. 109.

[5]. R. E. Sorace, V. S. Reinhardt, and S. A. Vaughn, "High-speed digital-to-RF converter," U.S. Patent 5 668 842, Sept. 16, 1997.

[6]. (2002) The IEEE website. [Online]. Available: http://www.ieee.org/

[7]. M. Shell. (2002) IEEEtran homepage on CTAN. [Online]. Available: http://www.ctan.org/tex-archive/macros/latex/contrib/supported/IEEEtran/

[8]. FLEXChip Signal Processor (MC68175/D), Motorola, 1996.

[9]. "PDCA12-70 data sheet," Opto Speed SA, Mezzovico, Switzerland.

[10]. A. Karnik, "Performance of TCP congestion control with rate feedback: TCP/ABR and rate adaptive TCP/IP," M. Eng. thesis, Indian Institute of Science, Bangalore, India, Jan. 1999.

[11]. J. Padhye, V. Firoiu, and D. Towsley, "A stochastic model of TCP Reno congestion avoidance and control," Univ. of Massachusetts, Amherst, MA, CMPSCI Tech. Rep. 99-02, 1999.

[12]. Wireless LAN Medium Access Control (MAC) and Physical Layer (PHY) Specification, IEEE Std. 802.11, 1997.