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Integral solutions of Quadratic Diophantine equation $10w^2 - x^2 - y^2 + z^2 = t^2$ with five unknowns

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ABSTRACT: The quadratic Diophantine equation given by $10w^2 - x^2 - y^2 + z^2 = t^2$ is analyzed for its patterns of non-zero distinct integral solutions. A few interesting relations between the solutions and special polygonal numbers are exhibited.
Keywords: Ternary quadratic, integral solutions, polygonal numbers.

I. INTRODUCTION

Quadratic equations are rich in variety [1-3].For an extensive review of sizable literature and various problems, one may refer [1-16]. In this communication, we consider yet another interesting quadratic equation $10w^2 - x^2 - y^2 + z^2 = t^2$ and obtain infinitely many non-trivial integral solutions.
A few interesting relations between the solutions and special Polygonal numbers are presented.

Notations Used

- $t_{m,n}$ - Polygonal number of rank 'n' with size 'm'
- CP_n^6 - Centered hexagonal Pyramidal number of rank n
- Gno_n – Gnomic number of rank 'n'
- FN_A^4 – Figurative number of rank 'n' with size 'm'
- Pr_n - Pronic number of rank 'n'
- P_n^m - Pyramidal number of rank 'n' with size 'm'
- ky_n - Keynea number

II. METHODS OF ANALYSIS

The Quadratic Diophantine equation with five unknowns to be solved for its non zero distinct integral solutions is

$$10w^2 - x^2 - y^2 + z^2 = t^2 \quad (1)$$

On substituting the linear transformation

$$\left.\begin{aligned} x &= w + z \\ y &= w - z \end{aligned}\right\} \quad (2)$$

in (1), it leads to

$$8w^2 - t^2 = z^2 \quad (3)$$

We obtain different patterns of integral solutions to (1) through solving (3) which are illustrated as follows:

Pattern I

Equation (3) can be written as

$$4w^2 - t^2 = z^2 - 4w^2$$

$$\frac{2w + t}{z + 2w} = \frac{z - 2w}{2w - t} = \frac{A}{B} \quad , B \neq 0$$

Considering

$$\frac{z - 2w}{2w - t} = \frac{A}{B}$$

Which is equivalent to the system of equations

$$\left.\begin{array}{r} w = w(A,B) = A^2 + B^2 \\ \mathrm{t} = \mathrm{t(A,B)} = 2\mathrm{A}^2 + 4\mathrm{AB} - 2\mathrm{B}^2 \\ \mathrm{z} = \mathrm{z(A,B)} = 2\mathrm{B}^2 + 4\mathrm{AB} - 2\mathrm{A}^2 \end{array}\right\} \qquad (4)$$

Substituting $(4)$and (3)in (2), the corresponding non-zero distinct integral solutions of (1) are given b

$$\left.\begin{array}{r} \mathrm{x(A,B)} = \mathrm{x} = 3\mathrm{B}^2 - \mathrm{A}^2 + 4\mathrm{AB} \\ \mathrm{y(A,B)} = \mathrm{y} = 3\mathrm{A}^2 - \mathrm{B}^2 - 4\mathrm{AB} \\ \mathrm{z(A,B)} = \mathrm{z} = 2\mathrm{B}^2 + 4\mathrm{AB} - 2\mathrm{A}^2 \\ \mathrm{w(A,B)} = \mathrm{w} = \mathrm{A}^2 + \mathrm{B}^2 \\ \mathrm{t(A,B)} = \mathrm{t} = 2\mathrm{A}^2 + 4\mathrm{AB} - 2\mathrm{B}^2 \end{array}\right\} \qquad (5)$$

Properties

1. $x(A-1) + 3y(A-1) - 16\, t_{3,A} \equiv 0$
2. $2x(1,B) + t(1,B) - 8\, t_{3,A} \equiv 0 \ (mod\ 8)$
3. $t(A,1) + 2w(A,1) - 8\, t_{3,A} \equiv 0$
4. $t(A(A+2),1) + 2w(A(A+2),1) - 24\, P_A^3 \equiv 0$
5. $t(A(A+2)(A+3),1) + 2w(A(A+2)(A+3),1) - 96\, P_A^4 \equiv 0$
6. $t(A(A+1)(A+2),1) + 2w(A(A+1)(A+2),1) - 48\, F_{4,n-4} \equiv 0$
7. $z(1,B) - 4\, t_{3,B} \equiv -2(mod\ \ 2)$
8. $2x(1,B) + t(1,B) - \ t_{8,B} = 0(mod\ 12)$
9. $x(A,1) + y(A,1) - t_{4,A} \equiv 2\ (mod\ \ 3)$
10. $x(A,A) + y(A,A)$ can be expressed as a perfect square.
11. Each of the following expressions represents a perfect number
    - (i) $[t(2,1) + z(1,2)]$
    - (ii) $[x(2,1) + x(2,2)]t(2,2)]$
    - (iii) $x(1,1)$
    - (iv) $t(1,2) + y(1,1)$
    - (v) $x(2,2) + z(1,1)$
12. Each of the following expressions represents a perfect square
    - (i) $w(3,3) + x(2,1)$
    - (ii) $x(3,3) + x(2,2) + y(2,1)$
    - (iii) $2x(3,3) + x(1,2) + y(2,1)$
13. Each of the following expressions represents a cube number
    - (i) $x(3,3) + x(1,2)$
    - (ii) $t(1,2)$
    - (iii) $z(1,2) + w(1,2) + t(1,2)$
    - (iv) $x(3,3) + 2z(3,3) - \frac{1}{2}w(1,1)$
    - (v) $4x(3,3)$
14. Each of the following expressions represents nasty number
    - (i) $x(2,2) + x(1,1)$
    - (ii) $x(1,1)x(2,2)$
    - (iii) $x(1,1)w(1,2)$
    - (iv) $y(2,1)z(2,1)w(2,1)$
    - (v) $x(2,2)w(1,2)$

Pattern II

The equation (3) can be written as

$$8w^2 - t^2 = z^2 * 1 \qquad (6)$$

Then write

$$\left.\begin{array}{r} z = 8a^2 - b^2 \\ 1 = (\sqrt{2}+1)(\sqrt{2}-1) \end{array}\right\} \qquad (7)$$

Substituting (7) into (6) reduces to

$$8w^2 - t^2 = (2\sqrt{2}\,w + t)(2\sqrt{2}\,w - t)$$

Equating rational and irrational parts, the coefficient values are

$$\left.\begin{aligned} w = w(a,b) &= 4a^2 + \frac{b^2}{2} + 2ab \\ t = t(a,b) &= 8a^2 + b^2 + 8ab \\ z = \mathrm{z(a,b)} &= 2\mathrm{b}^2 + 4\mathrm{ab} - 2\mathrm{a}^2 \end{aligned}\right\} \quad (8)$$

As our interest is on finding integer solutions, it is seen that the values of x, y and z are integers when both a and b are of the same parity. Thus by taking $a = 2A$, $b = 2B$ in (7) and substituting the corresponding values of u, v in (2) the non-zero integral solutions of (1) are given by

$$\left.\begin{aligned} x = x(A,B) &= 12A^2 - 2B^2 + 4AB \\ y = y(A,B) &= -4A^2 + 6B^2 + 4AB \\ z = z(A,B) &= 8A^2 - 4B^2 \\ w = w(A,B) &= 4A^2 + 2B^2 + 4AB \\ t = t(A,B) &= 8A^2 + 4B^2 + 16AB \end{aligned}\right\} \quad (9)$$

Properties

1. $w(A,B) - y(A,B) = z(A,B)$
2. $z(A,1) + t(A,1) - 16Pr_A \equiv 0$
3. $w(A,1) - y(A,1) - 8\,t_{4,A} + 4 \equiv 0$
4. $x(1,B) + w(1,B) - 4Gn_0 - 20 \equiv 0$
5. $2y(1,B) + z(1,B) - 16t_B \equiv 0$
6. $w(A,1) + t(A,1) - 24t_A \equiv 6(mod\ 8)$
7. $x(1,B) + 3y(1,B) - 32\,t_{8,B} = 0$
8. $x(1,B(B+2)) + 3y(1,B(B+2)) - 96\,P_A^3 \equiv 0$
9. $x(1,B^2(B+1)) + 3y(1,B^2(B+1)) - 32\,P_A^5 \equiv 0$
10. $x(1,B(B+2)(B+3)) + 3y(1,B(B+2)(B+3)) - 384\,P_A^4 \equiv 0$
11. Each of the following expressions represents a Nasty number
    (i) $[t(1,3) + z(1,1)]$
    (ii) $[t(2,3) + z(2,2)]$
    (iii) $x(3,3) - x(1,3)$
    (iv) $x(3,1) + w(1,2) + x(1,0)$
    (v) $w(1,1) + x(1,0) + z(1,0)$
    (vi) $t(1,1) - z(1,1)$
12. Each of the following expressions represents a perfect number
    (i) $t(3,3) + 2x(3,2) - z(1,1)$
    (ii) $4t(3,1)$
    (iii) $8[t(3,3) + t(2,3) + t(3,2) + t(3,1) + x(3.1) + x(3,3)] + z(2,2)y(2,2)$
    (iv) $t(1,2) - w(1,2)$
    (v) $z(2,2) - w(1,1)$
13. Each of the following expressions represents a cube number
    (i) $t(1,3) + t(3,1)$
    (ii) $t(3,3) + t(3,2) + z(3,2) + w(1,2)$
    (iii) $4t(3,3) - z(1,0)$
    (iv) $t(3,3) - z(3,3)$
    (v) $z(3,1) - w(1,0)$
14. Each of the following expressions represents perfect square
    (i) $w(2,2) + y(2,2)$
    (ii) $t(3,3) + t(0,1)$
    (iii) $x(3,1) + z(2,2) + w(1,1)$
    (iv) $2t(3,2) + t(2,3) + t(3,3)$
    (v) $y(2,3) + w(0,1)$

Pattern III

Consider the linear transformation

$$w = u - v, \quad t = 2u - 4v \tag{10}$$

Substituting (9) into (3) the equation reduces to

$$\frac{2u+z}{8v} = \frac{v}{2u-z} = \frac{A}{B}, \quad B \neq 0$$

Simplifying and employing the method of factorization,

$u = 8A^2 + B^2$
$v = 4AB$
$z = 3A^2 + B^2$

Substituting the corresponding values of u,v in (2) the non-zero integral solutions of (1) are given by

$$\left.\begin{aligned} x(A,B) = x &= 24A^2 - B^2 - 4AB \\ y(A,B) = y &= 3B^2 - 8A^2 - 4AB \\ z(A,B) = z &= 16A^2 - 2B^2 \\ w(A,B) = w &= 8A^2 + B^2 - 4AB \\ t(A,B) = t &= 16A^2 + 2B^2 - 16AB \end{aligned}\right\} \tag{11}$$

Properties

1. $z(A,1) - t(A,1) \equiv -4(mod\ 16)$
2. $y(1,B) + w(1,B) - 4\,Pr_A \equiv 0(mod\ 12)$
3. $3w(1,B)) - y(1,B) + 9Gn_0 + 41 \equiv 0$
4. $y(A,A^2) + w(A,A^2) + 8CP_A^6 - 4t_{4,A} \equiv 0$
5. $x(B,B) - w(B,B) - 14t_{4,B} \equiv 0$
6. Each of the following expressions represents a Nasty number
   (i) $[2x(1,2)]$
   (ii) $[t(2,1) - w(1,2)]$
   (iii) $x(2,3)$- x(1,3)
   (iv) $x(3,2) - z(1,2)$
   (v) $x(3,3) - 3y(1,3)$
7. Each of the following expressions represents a perfect number
   (i) $w(2,2) + t(2,2)$
   (ii) $z(3,1) + x(3,2) + x(3,1) + y(2,1)$
   (iii) $2x(3,1) + x(3,2) + w(2,2) + y(1,3)$
   (iv) $y(1,2)$
   (v) $z(1,1) - z(1,2)$
8. Each of the following expressions represents a cube number
   (i) $x(3,2) - y(1,2)$
   (ii) $z(2,2) + t(2,2)$
   (iii) $x(3,1) + z(3,1)$
   (iv) $2z(3,1) + x(3,1) + w(2,2) + w(1,1)$
   (v) $3z(3,1) + 2[x(3,1) - y(3,1) + w(1,2)]$
9. Each of the following expressions represents perfect square
   (i) $x(2,1) - y(2,1) + w(2,2)$
   (ii) $x(3,2) + z(3,2)$
   (iii) $x(3,1) - y(1,1) + x(3,2)$
   (iv) $z(3,3) - w(1,1)$
   (v) $t(3,3) + y(1,3)$

Pattern IV

Consider the transformation

$$(2u+z)(2u-z) = 8 * v^2 \tag{12}$$

Equating positive and negative factor, we get

$$(2u+z) = v^2$$
$$(2u-z) = 8$$

Solving the above equations, we get

$$\left.\begin{aligned} u &= \frac{v^2+8}{4} \\ z &= \frac{v^2-8}{2} \end{aligned}\right\} \quad (13)$$

As our interest on finding integer solutions, choose v so that u and z are integers
Then write

$$\left.\begin{aligned} u &= A^2+2A+3 \\ v &= 2A+2 \end{aligned}\right\} \quad (14)$$

Substituting (14) in (13), the solutions are

$$\left.\begin{aligned} x(A,B) &= x = 3A^2+4A-1 \\ y(A,B) &= y = -A^2-4A+3 \\ z(A,B) &= z = 2A^2+4A-2 \\ w(A,B) &= w = A^2+1 \\ t(A,B) &= t = 2A^2-4A-2 \end{aligned}\right\} \quad (15)$$

Properties

1. $w(2) - y(7)$ is a Kynea number
2. $y(A) - x(A) + 2z(A) = 0$
3. $y(A^2) + w(A^2) + 4CP_A^6 \equiv 4$
4. $z(A) + 2w(A) - 2Gn_0 \equiv 2$
5. Each of the following expression represents a perfect Square
   (i) $z(9)$
   (ii) $x(7) + y(7)$
   (iii) $z(9) + t(9)$
   (iv) $x(9) + z(9) + 2t(3)$
   (v) $x(8) + x(10) + t(7) - t(3)$
6. Each of the following expression represents a perfect number
   (i) $[x(9) + x(8) - t(3)]$
   (ii) $z(3)$
   (iii) $2[x(8)] + w(7)$
   (iv) $z(7) + y(7) = w(7)$
7. Each of the following expression represents a cube number
   (i) $x(10) + z(1)$
   (ii) $x(10) + z(10) - w(8)$
   (iii) $x(9) + w(8)$
   (iv) $w(6) + z(4) - y(5)$
   (v) $x(8) + x(10) - w(7)$
8. Each of the following expression represents a nasty number
   (i) $[w(8) - w(2)]$
   (ii) $[z(7) - 2w(1)]$
   (iii) $w(10) - w(2)$
   (iv) $x(3) + z(4) + x(5) + w(1)$
   (v) $x(7) - 6z(1)$

III. CONCLUSION

To conclude, one may search for other patterns of solutions and their corresponding properties.

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