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# Hankel determinants of RNA secondary structure sequences

Yu-Fang Xia[1], Yu-Qing Jin[2], Guo-Wei Jia[3]
[1,2,3]*Qufu Normal University, Department of Mathematics, Qufu，273165,PR China.*
*Corresponding Author: Yu-Fang Xia[1],*

**ABSTRACT:-** Let $(S_{n+1}^l)_{n\geq 0}$ be secondary structure sequences, $S_n^l$ satisfy the following recurrence $S_{n+1}^l = S_n^l + \sum_{k=l}^{n-1} S_k^l S_{n-1-k}^l$ for $n \geq l+1$ with $S_0^l = S_1^l = \cdots = S_{l+1}^l = 1$. In particular, when $l=0$, $S_n^0 = M_n$, where $M_n$ are the Motzkin numbers. We evaluate Hankel determinant $\det(S_{i+j+l}^l)_{0\leq i,j\leq n}$ and $\det(S_{i+j+l+1}^l)_{0\leq i,j\leq n}$. We show that $H_{n+1}^l = (S_{i+j+l+1}^l)_{0\leq i,j\leq n} = 1$ for $l \geq 0$. We give another proof of $\det(M_{i+j+1})_{0\leq i,j\leq n}$.

**Keywords:-** RNA secondary structure; Hankel determinant;Motzkin numbers; enumeration; sequence

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## I. INTRODUCTION

The secondary structure is a subset of helical structure consistent with a planar graph. The tertiary structure is the non-planar folding. Secondary and tertiary structures determine the three-dimensional shape of RNA molecule, hence determine the function of these important biological molecules [9]. Given a sequence $(a_n)_n \geq 0$, define its $(n+1)\times(n+1)$ Hankel determinant $\det(a_{i+j})_{0\leq i,j\leq n}$. Hankel determinants occur naturally in diverse areas of mathematics, we refer the readers to krattenthaler [7] for an excellent survey of determinant evaluations and [8] for a complement. We state some results needed for our study [6,10]. For a fixed integer $l \geq 0$, $S_{n+1}^l$ satisfy the following recurrence $S_{n+1}^l = S_n^l + \sum_{k=l}^{n-1} S_k^l S_{n-1-k}^l$, $S_0^l = S_1^l = \cdots = S_{l+1}^l = 1$ ,where $(S_{n+1}^l)_{n\geq 0}$ are secondary structure sequences. The sequence $(a_n)_n \geq 0$ of positive numbers is log-convex if the sequence $a_n/a_{n-1}$ is increasing. The sequences $(S_n^l)_{n\geq 0}$ are log-convex, for $l = 1,2,3$ and $4$, see [4]. In particular, when $l=0$, $S_n^0 = M_n$, where $M_n$ are the Motzkin numbers. The Motzkin number sequence $1,1,2,4,9,21,51,\cdots$, enumerates many different combinatorial objects and has been the subject of several studies [1]. Hankel determinants of the Motzkin numbers have occurred in the literature [1,2,3,5]. We evaluate two families of Hankel determinants. $H_{n+1}^l = \det(S_{i+j+l}^l)_{0\leq i,j\leq n}$, $H_{n+1}^{l+1} = \det(S_{i+j+l+1}^{l+1})_{0\leq i,j\leq n}$. In particular, when $l=0$, $H_{n+1}^0 = \det(M_{i+j})_{0\leq i,j\leq n}$. This determinant evaluation is already known. Cameron and Yip [3] evaluate this Hankel determinant by using combinational method. We give another proof of this determinant.

## II. THEOREM AND PROPOSITION

### A. Theorem

$H_{n+1}^l = (S_{i+j+l+1}^l)_{0\leq i,j\leq n} = 1$ for $l \geq 0$,where $S_{n+1}^l$ satisfy the recurrence.

### B. Proposition

$$H_{n+1}^{1}=\det(M_{i+j+1})_{0\le i,j\le n}=\begin{cases}1 & n\equiv 0,5 \pmod 6\\ 0 & n\equiv 1,4 \pmod 6\\ -1 & n\equiv 2,3 \pmod 6\end{cases}$$

## III. PROOF

All determinants in this paper will be evaluated by employing row, column operations and using the recursive relation of sequence.

### A. Proof of Theorem

$$H_{n+1}^{l}=\begin{vmatrix} S_l^l & S_{l+1}^l & \cdots & S_{l+n}^l\\ S_{l+1}^l & S_{l+2}^l & \cdots & S_{l+n+1}^l\\ \vdots & \vdots & \ddots & \vdots\\ S_{l+n}^l & S_{l+n+1}^l & \cdots & S_{l+2n}^l\end{vmatrix}_{n+1}$$

1) Subtract the $i$th column multiplied by $S_{n-1-i}^{l}$ from the $(n+1)$th column for $i=1,2,3,\cdots n-1$ and subtract the nth column. Next, applying a similar procedure to the $n$th, $(n-1)$th, $\cdots$, $2$nd columns by using recursion relation, we have

$$\begin{vmatrix} S_l^lS_0^l & S_l^lS_1^l & \cdots & S_l^lS_{n-1}^l\\ \sum_{i=l}^{l+1}S_i^lS_{l+1-i}^l & \sum_{i=l}^{l+1}S_i^lS_{l+2-i}^l & \cdots & \sum_{i=l}^{l+1}S_i^lS_{l+n-i}^l\\ \vdots & \vdots & \ddots & \vdots\\ \sum_{i=l}^{l+n-1}S_i^lS_{l+n-1-i}^l & \sum_{i=l}^{l+n-1}S_i^lS_{l+n-i}^l & \cdots & \sum_{i=l}^{l+n-1}S_i^lS_{l+2n-2-i}^l\end{vmatrix}_{n}$$

2) Perform row operations for the above determinant. For fixed $k=2,3,\cdots,n$, we subtract $i$ th row multiplied by $S_{k+l-i}^{l}/S_l^l$, for $i=1,2,\cdots,k-1$, from the $k$th row.

$$\begin{vmatrix} S_0^l & S_1^l & \cdots & S_{n-1}^l\\ S_1^l & S_2^l & \cdots & S_n^l\\ \vdots & \vdots & \ddots & \vdots\\ S_{n-1}^l & S_n^l & \cdots & S_{2n-2}^l\end{vmatrix}_{n}$$

From the result above, we take $l=0$.

$$\begin{vmatrix} S_0^0 & S_1^0 & \cdots & S_n^0\\ S_1^0 & S_2^0 & \cdots & S_{n+1}^0\\ \vdots & \vdots & \ddots & \vdots\\ S_n^0 & S_{n+1}^0 & \cdots & S_{2n-2}^0\end{vmatrix}_{n+1}$$

Then we get

$H_{n+1}^{l}=H_n^0=\cdots=H_1^0=S_0^0=1$.

3)Next let's consider

$$H_{n+1}^{l+1}=\begin{vmatrix} S_{l+1}^{l} & S_{l+2}^{l} & \cdots & S_{l+n+1}^{l} \\ S_{l+2}^{l} & S_{l+3}^{l} & \cdots & S_{l+n+2}^{l} \\ \vdots & \vdots & \ddots & \vdots \\ S_{l+n+1}^{l} & S_{l+n+2}^{l} & \cdots & S_{l+2n+1}^{l} \end{vmatrix}_{n+1}$$

By using the recursion relation, applying the similar procedure above then we have

$$\begin{vmatrix} 1 & S_{0}^{l} & \cdots & S_{n-1}^{l} \\ S_{0}^{l} & S_{1}^{l} & \cdots & S_{n}^{l} \\ \vdots & \vdots & \ddots & \vdots \\ S_{n-1}^{l} & S_{n}^{l} & \cdots & S_{2n-1}^{l} \end{vmatrix}_{n+1}$$

In the general case, we can't get the interesting results. Noting that $S_{n+1}^{0}=M_{n+1}$, by using above results, employing row and column operations for the determinants and using the recursion relation, we get the proposition.

**B. Proof of Proposition**

$$\begin{vmatrix} 1 & M_{0} & \cdots & M_{n-1} \\ M_{0} & M_{1} & \cdots & M_{n} \\ \vdots & \vdots & \ddots & \vdots \\ M_{n-1} & M_{n} & \cdots & M_{2n-1} \end{vmatrix}_{n+1}$$

1)Subtract the $i$th column multiplied by $M_{n-1-i}$ from the $(n+1)$th column for $i=1,2,\cdots,k-1$ and subtract the $n$th column. Next, applying a similar procedure to the $n$th, $(n-1)$th, $\cdots$, 2nd columns by using the recursion relation, we have

$$\begin{vmatrix} 0 & M_{0}M_{0} & \cdots & M_{0}M_{n-2} \\ M_{0}M_{0} & \sum_{i=0}^{1}M_{i}M_{1-i} & \cdots & \sum_{i=0}^{1}M_{i}M_{n-1-i} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i=0}^{n-2}M_{i}M_{n-2-i} & \sum_{i=0}^{n-1}M_{i}M_{n-1-i} & \cdots & \sum_{i=0}^{n-1}M_{i}M_{2n-3-i} \end{vmatrix}_{n}$$

2) Perform row operations for the above determinant. For fixed $k=1,2,\cdots,n$,we subtract $i$th row multiplied by $M_{k-i}/M_{0}$, for $i=1,2,\cdots,k-1$, from the $k$th row, and $M_{0}=1$. We have

$$\begin{vmatrix} 0 & M_{0} & \cdots & M_{n-2} \\ M_{0} & M_{1} & \cdots & M_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ M_{n-2} & M_{n-1} & \cdots & M_{2n-3} \end{vmatrix}$$

3) Subtract the $i$th column multiplied by $M_{n-2-i}$ from the nth column for $i=2,\cdots,n-2$ and subtract the $(n-1)$th column. Next, applying a similar procedure to the $(n-1)$th, $\cdots$, $2$nd columns by using the recursion relation, we have

$$-\begin{vmatrix} M_0 & M_0M_0 & \cdots & M_0M_{n-3} \\ M_1 & \sum_{i=0}^{1} M_iM_{1-i} & \cdots & \sum_{i=0}^{1} M_iM_{n-2-i} \\ \vdots & \vdots & \ddots & \vdots \\ M_{n-2} & \sum_{i=0}^{n-2} M_iM_{n-2-i} & \cdots & \sum_{i=0}^{n-2} M_iM_{2n-5-i} \end{vmatrix}_{n-1}$$

4) For fixed $k=2,3,\cdots,n-1$, we subtract $i$th row multiplied by $M_{k-i}/M_0$, for $i=1,2,\cdots,k-1$, from the $k$th row. we have

$$-\begin{vmatrix} M_1 & M_2 & \cdots & M_{n-2} \\ M_2 & M_3 & \cdots & M_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ M_{n-2} & M_{n-1} & \cdots & M_{2n-5} \end{vmatrix}_{n-2}$$

then we can obtain $H_{n+1}^1 = -H_{n-2}^1$. This completes the proof of proposition.

## IV. FURTHERWORK

We evaluate Hankel determinants $\det(S_{i+j+l}^l + S_{i+j+l+1}^l)_{0\le i,j\le n}$ and $\det(S_{i+j+l+1}^l + S_{i+j+l+2}^l)_{0\le i,j\le n}$. Then, we prove the log-convexity by using a technique based on the Hankel determinants.


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## REFERENCES


[1]. M. Aigner, Motzkin numbers, European J. Combin 19 (6) (1998) 663-675.

[2]. M. Aigner, Caltalan and other numbers: a recurrent theme, in: H. Crapo, D. Senato (Eds.), Algebraic Combinations and Computer Science, Springer, Berlin, 2001, 347390.

[3]. N.T. Cameron, A.C. M. Yip, Hankel determinants of sums of consecutive Motzkin numbers, Linear Algebra Appl. 434 (3) (2011) 712-722.

[4]. T. Doslic, D. Svrtan, D. Veljan, Enumerative aspects of secondary structures, Discrete Math. 285 (2004) 67-82.

[5]. M. Elouafi, A unified approach for the Hankel determinants of classical combinatorial numbers, J. Math. Anal. Appl. 431 (2015) 1253-1274.

[6]. I.L. Hofacker, P. Schuster, P.F. Stadler, Combinatorics of RNA secondary structures, Discrete Math, 88 (1988) 207-237.

[7]. C. Krattenthaler, Advanced determinant calculus, Sem. Lothar. Combin. 42 (1999) Article B42q.

[8]. C. Krattenthaler, Advanced determinant calculus: a complement, Linear Algebra Appl. 411 (2005) 68-166.

[9]. W.R. Schmitt and M.S. Waterman, Linear trees and RNA secondary structure, Discrete Appl. Math. 51 (1994) 317-323.

[10]. P.R. Stein, M.S. Waterman, on some new sequences generalizing the Catalan and Motzkin numbers, Discrete Math, 26 (1978) 261-272.


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