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Odd Graceful Labeling of Few Graphs

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Abstract
Odd graceful labeling[4] is to inject $f: V(G) \rightarrow \{0,1,2,..,(2q-1)\}$ *whenever the edge* ab *is identified the label*$|f(a)-f(b)|$, *so that the resulting edge labels are* $\{1,3,5,..,(2q-1)\}$.*In this paper, we showed that a few graphs such as twig diamond graphs without prime edge coupled with pendent edges, SSG coupled with star graph, super subdivision of comb graphand bistar graph are odd graceful.*
Keywords:*Odd graceful labeling, Star graph, Twig diamond Graph, Comb graph, Sub-divided shell graph, Super subdivision.*
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I. INTRODUCTION

One of the most valuable technique in graph theory is graph labelling. It is defined as the process of allocating values to points, lines or both with certain constrains. Rosa [9] innovated the use of graceful labeling notation in 1967. Later, a few labeling techniques have been introduced, one of which was odd graceful, invented by Gnanajothi [4] in 1991. An odd graceful labeling is injecting f from $V(G)$ to $\{0,1,2,..,(2q-1)\}$ so that, when each edge ab is allocated the label $|f(a)-f(b)|$, the resulting edge labels $\{1,3,5,..,(2q-1)\}$are distinct The shell graph was first introduced by Deb and Limaye[2], and subdivided shell graphs are introduced by Jeba Jesintha[5]. Gnanajothi[4] proved the graphs the path P_n, the cycle C_n if and only if n is even, the complete bipartite graph $K_{m,n}$ are odd graceful.Badr E M [8], proved that crown graphs are odd graceful.

In this paper, we prove few graphs to be odd graceful such as twig diamond graphs without prime edge coupledwith pendent edges,SSG coupled with star graph, super subdivision of comb graph and bistar graph.

II. MAIN RESULTS

In this chapter we prove few definitions and theorems.

Definition 2.1.
The star graph [1]$S_n$of order n, sometimes simply known as an "n star graph" is a tree on n vertices with one vertex of degree $n-1$ and the other $n-1$ vertices each of degree 1.

Definition 2.2.
The diamond graph [6] is a planar undirected graph with 4 vertices and 5 edges. It consists of a complete graph K_4 minus one edge.

Definition 2.3.
The twig diamond graph [6]is a planar undirected graph with 8 vertices and 11 edges obtained by the attachment of two diamond graphs by an edge.

Definition 2.4.
Define a cycle C_n with $(n-3)$ chords with a common end point are called as apex is defined to be the shell graph [2]. Shell graphs are represented as $C(n, n-3)$. Shell graphs are also known as fan graph.In the shell graph,if each path is subdivided, we get a subdivided shell graph [5].

Definition 2.5.
Super subdivision of graph $S^*(G)$[7] is the graph obtainedfrom G by replacing every edge uvbya complete bipartite graph$K_{2,m}$.

Definition 2.6.
Comb [1] is a graph obtained by joining a single pendant edge to each vertex of a path.

Definition 2.7.
Bistar $B_{n,n}$[1] is the graph obtained by joing the apex vertices of two copies of $K_{1,n}$ by an edge. The vertex set of $B_{n,n}$ is $V(B_{n,n}) = \{u, v, u_i, v_i: 1 \le i \le n\}$ where u and v are apex vertices and $u_i, v_i$are pendant vertices.

Theorem 2.8.
Twig diamond graph without prime edge coupled with pendent edges admits odd graceful labeling.
Proof.
The graph H is described as follows: Let us consider the two diamond graphs without middle edge as $H_1$and H_2 having the vertices$a_1, a_2, a_3, a_4$and a_5, a_6, a_7, a_8 in the anti-clockwise direction respectively. The graph $H_1$and H_2 attached by the pendent edge joininga_4 anda_6. The remaining vertices of the graphs $H_1$and H_2attached with the pendent edges. The pendent edges attached to $a_1$be denoted as $u_1, u_2, u_3, \ldots, u_r$the pendent edges attached to a_2 is denoted as $v_1, v_2, v_3, \ldots, v_r$and the pendent edges attached to a_3 is denoted as $w_1, w_2, w_3, \ldots, w_r$ in the anti-clockwise direction. Similarly, the pendent edges attached to a_5 is denoted as $x_1, x_2, x_3, \ldots, x_r$the pendent edges attached to a_7 is denoted as $z_1, z_2, z_3, \ldots, z_r$ and the pendent edges attached to a_8 is denoted as $y_1, y_2, y_3, \ldots, y_r$. The resultant graph is called as twig diamond graph without middle edge attached pendent edges and shown in Figure 1.
The graph H has $V(G) = 6r + 8, E(G) = 6r + 9$.
Vertex labeling for the twig diamond graph without middle edge are:

$$\beta(a_1) = 0; \beta(a_3) = 4;$$
$$\beta(a_2) = 12r + 17;\ \beta(a_4) = 12r + 15$$
$$\beta(a_5) = 12r + 13;\ \beta(a_7) = 12r + 11$$
$$\beta(a_6) = 6r + 6;\ \beta(a_8) = 6r + 10$$

Vertex labeling for the pendent edges is:

$$\beta(u_i) = 12r + 11 - 2i; 1 \le i \le r$$
$$\beta(v_i) = 2r + 6 + 2i; 1 \le i \le r$$
$$\beta(w_i) = 8r + 15 - 2i; 1 \le i \le r$$
$$\beta(x_i) = 6r + 12 + 2i;\ 1 \le i \le r$$
$$\beta(y_i) = 2r + 9 + 2i; 1 \le i \le r$$
$$\beta(z_i) = 10r + 10 + 2i;\ 1 \le i \le r$$

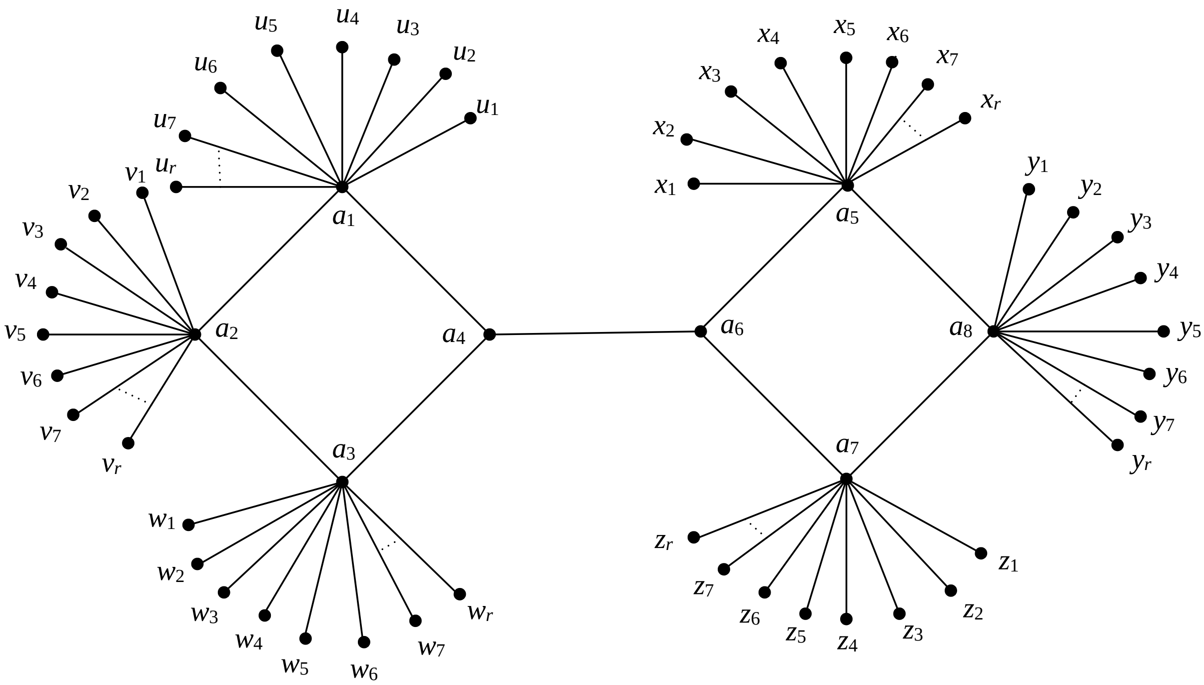

Figure 1. The graph H

Edge labeling are calculated as follows:

$$E_1 = |\beta(a_1) - \beta(u_i)| = |12r + 11 - 2i|;\ 1 \le i \le r$$
$$E_2 = |\beta(a_3) - \beta(w_i)| = |\ 8r + 11 - 2i|;\ 1 \le i \le r$$
$$E_3 = |\beta(a_2) - \beta(v_i)| = |\ 10r + 11 - 2i|;\ 1 \le i \le r$$
$$E_4 = |\beta(a_4) - \beta(a_6)| = |6r + 9|$$

$$E_5 = |\beta(a_1) - \beta(a_2)| = |12r + 17|$$
$$E_6 = |\beta(a_1) - \beta(a_4)| = |12r + 15|$$
$$E_7 = |\beta(a_2) - \beta(a_3)| = |12r + 13|$$
$$E_8 = |\beta(a_3) - \beta(a_4)| = |12r + 11|$$
$$E_9 = |\beta(a_5) - \beta(x_i)| = |\, 6r + 1 - 2i|;\ 1 \le i \le r$$
$$E_{10} = |\beta(a_8) - \beta(y_i)| = |\, 4r + 1 - 2i|;\ 1 \le i \le r$$
$$E_{11} = |\beta(a_7) - \beta(z_i)| = |\, 2r + 1 - 2i|;\ 1 \le i \le r$$
$$E_{12} = |\beta(a_5) - \beta(a_6)| = |6r + 7|$$
$$E_{13} = |\beta(a_5) - \beta(a_8)| = |6r + 3|$$
$$E_{14} = |\beta(a_6) - \beta(a_7)| = |6r + 5|$$
$$E_{15} = |\beta(a_7) - \beta(a_8)| = |6r + 1|$$

Thus,

$$E(G) = E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5 \cup E_6 \cup E_7 \cup E_8 \cup E_9 \cup E_{10} \cup E_{11} \cup E_{12} \cup E_{13} \cup E_{14} \cup E_{15}$$

From the above labeling pattern, it is observed that the edge labels are distinct. Thus, the twig diamond graph without middle edge attached to pendent edges admits odd graceful labeling.

Theorem 2.9.
The star graph coupled with SSG admits odd graceful labeling.
Proof.

The graph G is obtained by coupling SSG with star graph. Let a be the apex vertex of the star graph. Let $b_i (1 \le i \le n)$ be the apex vertex of the SSG. Let $c_j^i (1 \le i \le n, 1 \le j \le m)$ be the

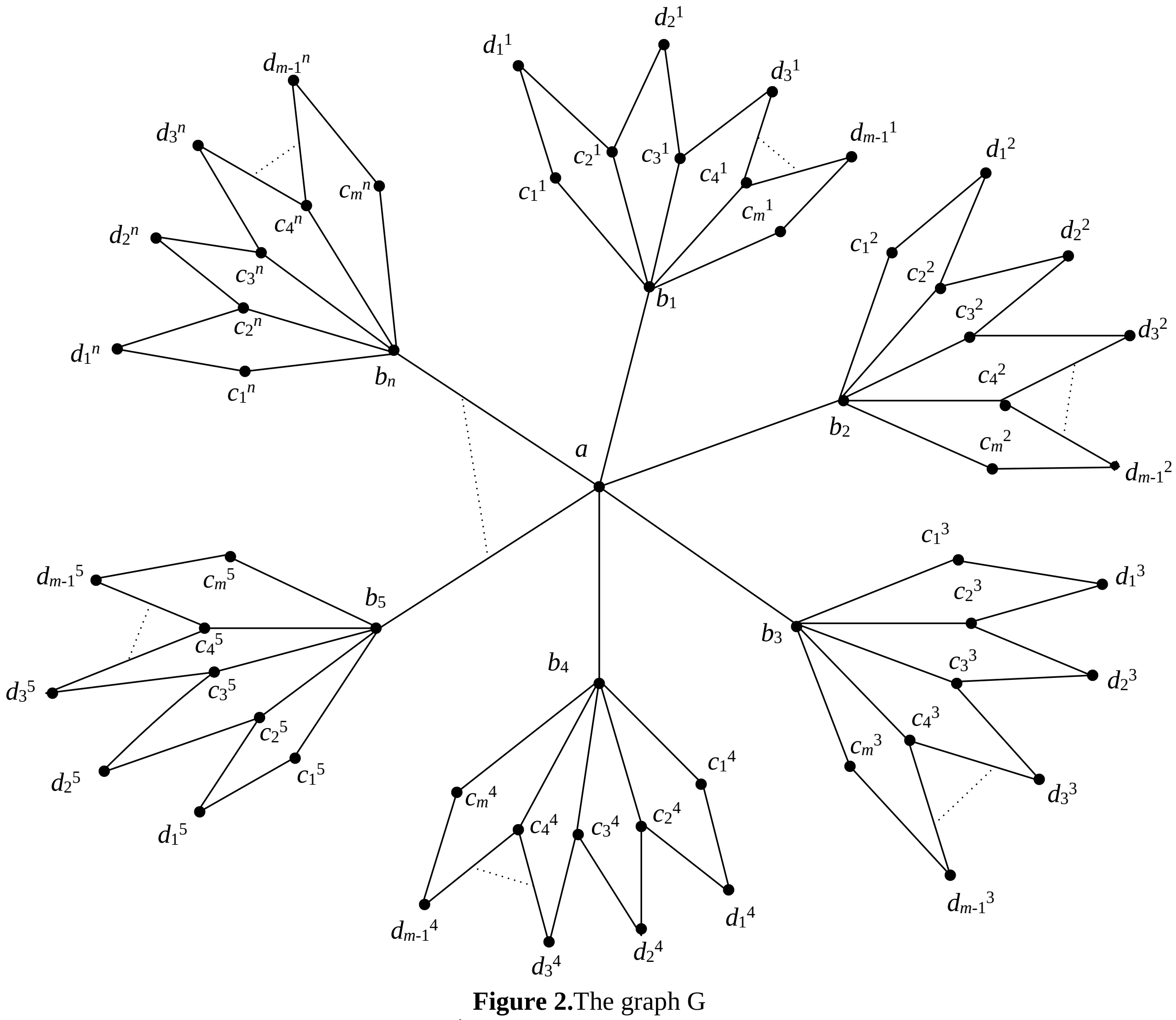

Figure 2. The graph G

vertices of the SSG adjacent to b_i and let $d_j^i (1 \le i \le n, 1 \le j \le m-1)$ be the vertices of the SSG not adjacent to b_i.

The graph G has $V(G) = 2nm + 1,\ E(G) = 3nm - n$.
Vertex labeling for the graph is defined as:

$$g(a) = 0$$
$$g(b_i) = 6nm - 2n - 2i + 1;\ 1 \le i \le n$$
$$g\left(c_j^{2i-1}\right) = 2mn + 2n + 4m - 4mi - 4i - 2j + 4;\ \ 1 \le i \le n, 1 \le j \le m$$
$$g\left(c_j^{2i}\right) = 2mn + 2n + 2m - 4mi - 4i - 2j + 2;\ \ 1 \le i \le n, 1 \le j \le m$$

When n is even

$$g\left(d_j^{2i-1}\right) = 2mn + 2n + 2j - 8i + 5;\ 1 \le i \le \frac{n}{2};\ 1 \le j \le m - 1$$
$$g\left(d_j^{2i}\right) = 4mn - 2m + 2j - 8i + 3;\ 1 \le i \le \frac{n}{2};\ 1 \le j \le m - 1$$

When n is odd

$$g\left(d_j^{2i-1}\right) = 4mn - 2m + 2j - 8i + 7;\ 1 \le i \le \frac{n-1}{2};\ 1 \le j \le m - 1$$
$$g\left(d_j^{2i}\right) = 2mn + 2n - 2m + 2j - 8i + 3;\ 1 \le i \le \frac{n-1}{2};\ 1 \le j \le m - 1$$

Edge labeling for the graph is defined as:

$$E_1 = |g(a) - g(b_i)| = |6nm - 2n - 2i + 1|;\ 1 \le i \le n$$
$$E_2 = \left|g(b_i) - g\left(c_j^{2i}\right)\right| = |4nm - 4n - 2m + 4mi + 2i + 2j - 1|;\ 1 \le i \le n, 1 \le j \le m$$
$$E_3 = \left|g(b_i) - g\left(c_j^{2i-1}\right)\right| = |4nm - 4n - 4m + 4mi + 2i + 2j - 3|;\ 1 \le i \le n, 1 \le j \le m$$

When $n$is even

$$E_4 = \left|g\left(c_j^{2i-1}\right) - g\left(d_j^{2i-1}\right)\right| = |\ 4m - 4mi + 4i - 4j - 1|;\ \ 1 \le i \le \frac{n}{2}, 1 \le j \le m - 1$$
$$E_5 = \left|g\left(c_j^{2i}\right) - g\left(d_j^{2i}\right)\right| = |2nm - 2n - \ 4m + 4mi - 4i + 4j - 1|; 1 \le i \le \frac{n}{2}, 1 \le j \le m - 1$$
$$E_6 = \left|g\left(d_j^{2i-1}\right) - g\left(c_{j+1}^{2i-1}\right)\right| = |\ 4mi - 4m + 4j - 4i + 3|; 1 \le i \le \frac{n}{2}, 1 \le j \le m - 1$$
$$E_7 = \left|g\left(d_j^{2i}\right) - g\left(c_{j+1}^{2i}\right)\right| = |2mn - 2n - \ 4m + 4mi - 4i + 4j + 7|;$$
$$1 \le i \le \frac{n}{2}, 1 \le j \le m - 1$$

When n is odd

$$E_8 = \left|g\left(c_j^{2i-1}\right) - g\left(d_j^{2i-1}\right)\right| = |\ 2nm - 2n - 6m + 4mi + 4j - 4i + 3|;$$
$$1 \le i \le \frac{n-1}{2}, 1 \le j \le m - 1$$
$$E_9 = \left|g\left(c_j^{2i}\right) - g\left(d_j^{2i}\right)\right| = |\ 4m - 4mi + 4i - 4j - 1|;\ 1 \le i \le \frac{n-1}{2}, 1 \le j \le m - 1$$
$$E_{10} = \left|g\left(d_j^{2i-1}\right) - g\left(c_{j+1}^{2i-1}\right)\right| = |\ 2nm - 2n - 6m + 4mi + 4j - 4i + 5|;$$
$$1 \le i \le \frac{n-1}{2}, 1 \le j \le m - 1$$
$$E_{11} = \left|g\left(d_j^{2i}\right) - g\left(c_{j+1}^{2i}\right)\right| = |2mn - 2n - \ 4m + 4mi - 4i + 4j + 3|;$$
$$1 \le i \le \frac{n-1}{2}, 1 \le j \le m - 1$$

$E(G) =\ E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5 \cup E_6 \cup E_7 \cup E_8 \cup E_9 \cup E_{10} \cup E_{11}$

From the above labeling pattern, it is observed that the edge labels $\{1,3,5,\ldots,(2q-1)\}$are distinct. Thus, the star coupled with SSG graph admits odd graceful labeling.

Theorem 2.10.

Super subdivision of comb graph is odd graceful.

Proof.

The graph H is obtained by the super subdivision of comb. The graph H is described as follows:
Let $u_i (1 \le i \le n)$ be the vertices of the path. Let $w_j^i (1 \le i \le n-1, 1 \le j \le m)$ be the vertices of the super subdivision of the path vertices. Let $v_i (1 \le i \le n)$ be the vertices of the pendant vertices of the comb. Let $x_j^i (1 \le i \le n, 1 \le j \le m)$ be the vertices of the super subdivision of pendent vertices of the comb.
The graph H has $V(H) = 2n + 2nm - m,\ E(H) = 4nm - 2m$.
Let the vertex labeling is defined as :

$$h(u_i) = 2i - 2;\ \ 1 \le i \le n$$

$$h(v_i) = 2nm + 2i - 2;\ \ 1 \le i \le n$$
$$h(w_i^j) = 8nm - 4mi - 4j + 2i + 1;\ \ 1 \le i \le n - 1, 1 \le j \le m$$
$$h(x_i^j) = 2nm - 2m + 2mi + 2j + 2i - 3;\ \ 1 \le i \le n, 1 \le j \le m$$

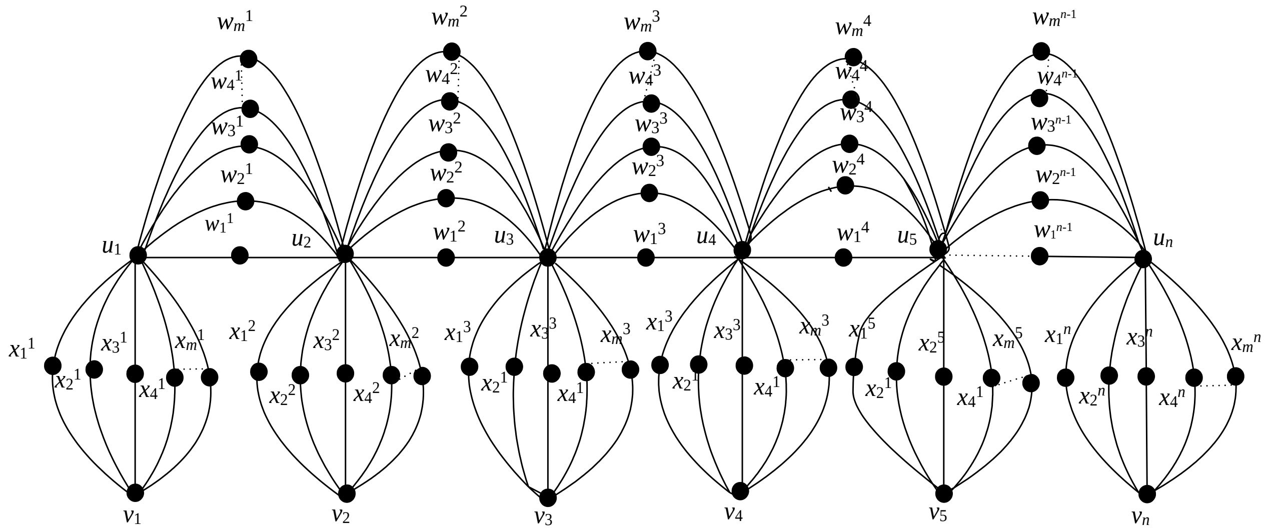

Figure 3. Super subdivision of comb graph

Let the edge labeling is defined as

$$E_1 = |h(u_i) - h(w_i^j)| = |8nm - 4mi - 4j + 3|;\ 1 \le i \le n - 1, 1 \le j \le m$$
$$E_2 = |h(w_i^j) - h(u_{i+1})| = |8nm - 4mi - 4j + 1|;\ 1 \le i \le n - 1, 1 \le j \le m$$
$$E_3 = |h(u_i) - h(x_i^j)| = |\ 2mn - 2m + 2mi + 2j - 1|;\ \ 1 \le i \le n, 1 \le j \le m$$
$$E_4 = |h(x_i^j) - h(v_i)| = |\ 2m - 2mi - 2j + 1|;\ \ 1 \le i \le n, 1 \le j \le m$$

Therefore, $E = E_1 \cup E_2 \cup E_3 \cup E_4$

From the above labeling pattern, it is observed that the edge labels $\{1,3,5,\dots,(2q-1)\}$are distinct. Thus, the super subdivision of comb graph admits odd graceful labeling.

Theorem 2.11.

Super subdivision of bistar is odd graceful.

Proof.

The super subdivision of bistar graph is denoted as G and the graph is described as follows: Let u be the apex vertex of the first star. Let v be the apex vertex of the second star. Let $u_j^i(1 \le i \le n)(1 \le j \le m)$be the vertices adjacent to the vertex u and let $v_j^i(1 \le i \le n)(1 \le j \le m)$be the vertices adjacent to the vertex v. Let$s_i(1 \le i \le n\ )$ be the vertices adjacent to u_j^i and let $t_i(1 \le i \le n\ )$ be the vertices adjacent to v_j^i. Let $w_j(1 \le j \le m\ )$ be the vertices of the super subdivision of the path joining the two stars. The graph is shown in figure 4. The number of vertices and number of edges is defined as $V(G) = 2(nm + n + 1), E(G) = 2m(2n + 1)$.

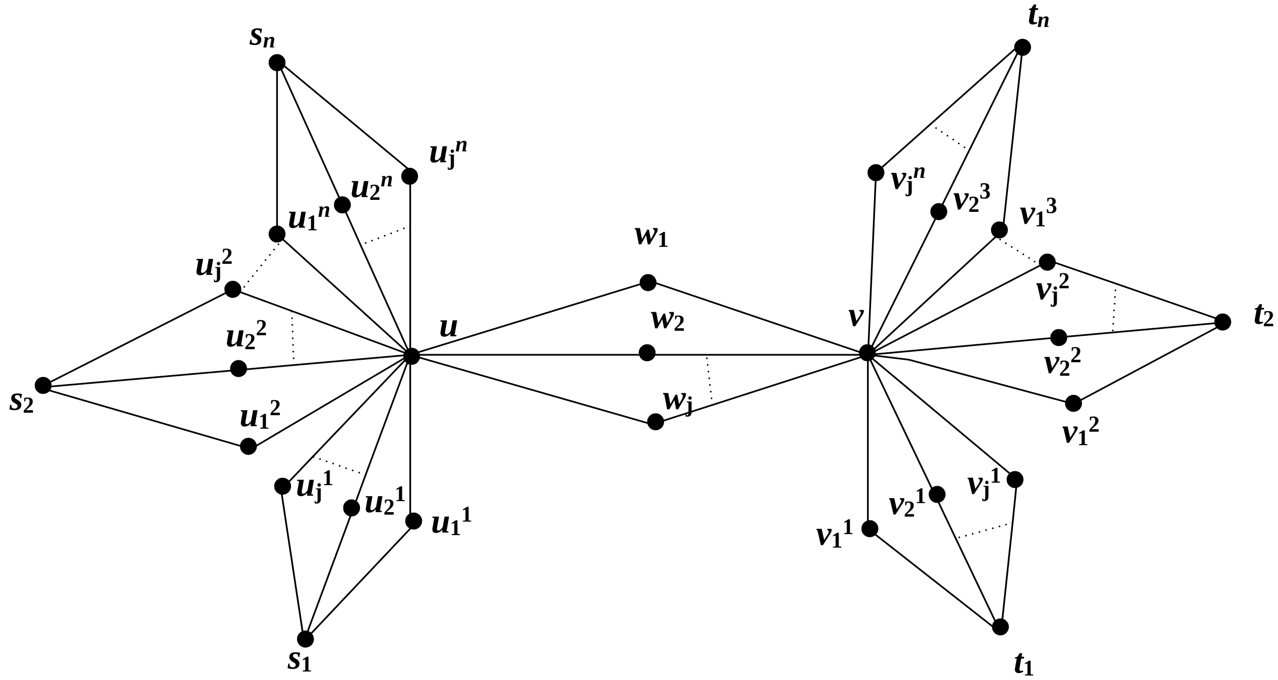

Figure 3. Super subdivision of bistar graph

Define the vertex labeling as:

$$f(u) = 0$$
$$f(v) = 4mn + 2m$$
$$f(t_i) = 4mi;\ (1 \le i \le n)$$
$$f(s_i) = 4mn - 4mi + 2m; (1 \le i \le n)$$
$$f(w_j) = 4mn + 2m + 2j - 1; (1 \le j \le m)$$
$$f(v_j^i) = 2mi - 2m + 2j - 1; (1 \le i \le n), (1 \le j \le m)$$
$$f(u_j^i) = 8mn + 6m - 2j - 2mi + 1;\ (1 \le i \le n), (1 \le j \le m)$$

Define the edge labeling as:

$$E_1 = |f(w_j) - f(v)| = 2j - 1;\ (1 \le j \le m)$$
$$E_2 = |f(u) - f(w_j)| = 4mn + 2m + 2j - 1; (1 \le j \le m)$$
$$E_3 = |f(u) - f(u_j^i)| = 8mn + 6m - 2j - 2mi + 1; (1 \le i \le n), (1 \le j \le m)$$
$$E_3 = |f(u_j^i) - f(s_i)| = 4mn + 2mi + 4m - 2j + 1; (1 \le i \le n), (1 \le j \le m)$$
$$E_3 = |f(v) - f(v_j^i)| = 4mn + 4m - 2mi - 2j + 1; (1 \le i \le n), (1 \le j \le m)$$
$$E_3 = |f(v_j^i) - f(t_i)| = 2mi + 2m - 2j + 1; (1 \le i \le n), (1 \le j \le m)$$

Therefore, $E = E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5 \cup E_6$

From the above labeling pattern, it is observed that the edge labels $\{1,3,5,\dots,(2q-1)\}$are distinct. Thus, the super subdivision of bistar graph admits odd graceful labeling.

III. CONCLUSION

In this paper, we prove few graphs to be odd graceful such as twig diamond graphs without prime edge coupled with pendent edges, Star graph coupled with SSG, Super subdivision of comb graph and bistar graph.

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