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Buckling Effect on Uniformly Clamped Thin Plate with Non Even Stiffness Coefficients, Under External Load

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Abstract
Different cases of supports systems were critically reviewed in this research. The work centered on uniformly supported plate. Odd energy functional was adopted. Considering two different rectangular thin isotropic plate elements, under the influence of external (load). The first case was simply supported all round while the other plate arrangement was Clamped all round. The shape functions, 3rd order strain energy equation and external load parameter were formulated from first principle after several minimization. The second case gave rise to the Third Order Overall Potential Energy Functional. Integrating the Total Energy functional with respect to the amplitude produced the Governing equation. Introduction of different coefficients explained the extent of their stiffnesses that were formulated. The Third order strain energy equation was also derived, and further minimization of the third order strain energy gave the Third Order Overall Potential Energy Functional. Next to this was the derivation of the critical buckling load equations which was gotten by further minimizing the governing equation. Aspect ratios of different values were substituted into the critical equation though this was done at different value interval. The non-dimensional buckling load parameters were obtained at arithmetic difference of 1/10. The results gotten in both cases were as detailed below.

Key word: *Third Order Overall Potential Energy, Strain Energy, Overall Energy potentials*

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THIRD ORDER ENERGY FUNCTIONAL FOR THE PLATES

The two uniformly supported plate under consideration in the research work are simple-simple-simple simple and clamped-clamped-clamped-clamped rectangular plates. The displacements of a thin rectangular plate include in-plane displacements – u and v and out of plane displacement– w. Considering u and v as the functions of x, y and z, w is only a function of x and y and so x, y and z are the principal coordinates. The implication of this is that w is constant along z direction. This is in consonant with the assumption that– "vertical normal strain of a plate is equal to zero". The vertical shear strains are negligible in classical plate analysis and assumed to be equal to zero. Thus, out of the six engineering strain components, ε_z , γ_{xz} and γ_{yz} were assumed to be zero. Therefore, leaving only three engineering strain components - ε_x, -ε_y, and γ_{xy}, upon the minimization of the strain deflection.

I. Methods

The stress, strain, shear tress, shear strain and the deflection were considered at different conditions, from the constitutive relations were formed. The strain energy was directly integrated to form the fundamental for the formulation of the needed functional. These properties were further introduced by substitution and this gave rise to the flexural rigidity. When the derived strain energy was added to the external work done, the overall potential energy, T_p was derived. The stages involved were as detailed below,

Having proved that, $\frac{\partial u}{\partial y} = -z\frac{\partial^2 w}{\partial x \partial y}$ 1

According to already established work,
and also that

$$\frac{\partial v}{\partial x} = -z\frac{\partial^2 w}{\partial x \partial y} \quad 2$$

That means

$$\gamma_{xy} = -z\frac{\partial^2 w}{\partial x\partial y} - z\frac{\partial^2 w}{\partial x\partial y} = -2z\frac{\partial^2 w}{\partial x\partial y} \qquad 3$$

and stress strain relationship, the stress deflection relationship were formulated

$$\sigma_x = \frac{-Ez}{1-\mu^2}(T_x) \qquad 4$$

where

$$T_x = \frac{\partial^2 w}{\partial x^2} + \mu\frac{\partial^2 w}{\partial y^2} \qquad 5$$

similarly

$$\sigma_y = \frac{-Ez}{1-\mu^2}(T_y) \qquad 6$$

$$T_y = \mu\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \qquad 7$$

The product of stress and strain at every point on the plate continuum were added together and that gave Equation 8

$$\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \tau_{xy}\gamma_{xy} = \sigma\varepsilon \qquad 8$$

Substituting and minimizing further gives the 3rd Strain energy equation as

$$E_u = \frac{D}{2}\int_0^a\int_0^b (E_{u1} + 2E_{u2} + E_{u3})\,dxdy \qquad 9$$

where

$$\frac{\partial^3 w}{\partial x^3}.\frac{\partial w}{\partial x} = E_{u1} \qquad 10$$

also

$$\frac{\partial^3 w}{\partial x\,\partial y^2}.\frac{\partial w}{\partial x} = E_{u2} \qquad 11$$

and

$$\frac{\partial^3 w}{\partial y^3}.\frac{\partial w}{\partial y} = E_{u3} \qquad 12$$

Introducing the external work done and the overall potential energy, gives

$$T_p = \frac{A^2 D}{2}\int_0^a\int_0^b (E_{uh1} + 2E_{uh2} + E_{uh3})\,dxdy - \frac{A^2 Nx}{2}\int\int\frac{\partial^2 h}{\partial x^2}.h\,dxdy \qquad 13$$

$$E_{uh1} = \frac{\partial^3 h}{\partial x^3}.\frac{\partial h}{\partial x}\ ,\ \frac{\partial^3 h}{\partial x\,\partial y^2}.\frac{\partial h}{\partial x}\ E_{uh2}\quad and\quad E_{uh3} = \frac{\partial^3 h}{\partial y^3}.\frac{\partial h}{\partial y} \qquad 14$$

Differentiating the Total Potential energy with respect to the Amplitude and further substitutions gives

$$B_{ct} = \frac{\frac{2AD}{2}\int_0^1\int_0^1 .\ (E_{uh1} + 2E_{uh2} + E_{uh3})dxdy}{\frac{2A}{2}\int_0^a\int_0^b\left(\frac{\partial h}{\partial x}\right)^2 dxdy} \qquad 15$$

further reducing Equation 15 gives

$$\mathrm{B_{ct}} = \frac{(\mathrm{ct_1} + 2\frac{1}{\mathrm{p^2}}\mathrm{ct_2} + \frac{1}{\mathrm{p^4}}\mathrm{ct_3})\frac{\mathrm{D}}{\mathrm{a^2}}}{\mathrm{ct_6}} \qquad 16$$

Where

The first buckling coefficient is

$$\mathrm{ct_1} = \int_0^1\int_0^1 (\mathrm{E}_{uh1})\mathrm{dRdQ} \qquad 17$$

The second buckling coefficient is

$$\mathrm{ct_2} = \int_0^1\int_0^1 (\mathrm{E}_{uh2})\mathrm{dRdQ} \qquad 18$$

The The third buckling coefficient is

$$\mathrm{ct_3} = \int_0^1\int_0^1 (\mathrm{E}_{uh3})\mathrm{dRdQ} \qquad 19$$

While the fourth critical buckling coefficient is

$$\mathrm{ct_4} = \int_0^1\int_0^1 (\frac{\partial \mathrm{h}}{\partial \mathrm{R}})^2\ \mathrm{dRdQ} \qquad 20$$

Buckling Load Equation for SSSS shape

The critical buckling load equation for SSSS can be written in terms of stiffness coefficients (ct_1, ct_2, ct_3 and ct_6) using the $a^2 = b^2/p^2$, for the .

aspect ratio of p = b/a as follows

$$\mathrm{B}_{crt} = \frac{\mathrm{D}\left(\mathrm{ct_1} + \frac{2}{\mathrm{p^2}}\mathrm{ct_2} + \frac{1}{\mathrm{p^4}}\mathrm{ct_3}\right)}{\mathrm{ct_6 a^2}} \qquad 21$$

l Buckling Load Equation for CCCC shape

Substituting the stiffness coefficients (ct_1, ct_2, ct_3 and ct_6) for the CCCC plate, into the general buckling load equation, the critical buckling load equation the plate can be expressed as

$$\mathrm{B}_{crt} = \frac{D(\mathrm{ct_1} + \frac{2}{p^2}ct_2 + \frac{1}{p^4}\mathrm{ct_3})}{\mathrm{k_6}a^2} \qquad 22$$

Considering $a^2 = b^2/p^2$, for p = b/a as the aspect ratio.

Determination of the Stiffness coefficients of the plates

From the Polynomial rules, the shape function of the shape function, sh for SSSS Plate

is as $(R-2R^3+R^4)(Q-2Q^3+Q^4)$ 23

Differential values for Simple-Simple-Simple-Simple shape

Tthe shape functions were differentiated at different levels and that gave

$$s\mathrm{h} = (R - 2\mathrm{R^3} + \mathrm{R^4})(\mathrm{Q} - 2\mathrm{Q^3} + \mathrm{Q^4}) \qquad 24$$

$$\frac{\partial \mathrm{sh}}{\partial \mathrm{R}} = (1 - 6\mathrm{R^2} + 4\mathrm{R^3})(\mathrm{Q} - 2\mathrm{Q^3} + \mathrm{Q^4}) \qquad 25$$

Differentiating Equation 25 with respect to R twice gives

$$\frac{\partial^2 sh}{\partial R^2} = (-12\mathrm{R} + 12\mathrm{R}^2)(\mathrm{Q} - 2\mathrm{Q}^3 + \mathrm{Q}^4) \qquad 26$$

Differentiating Equation 24 with respect to R thrice gives

$$\frac{\partial^3 sh}{\partial R^3} = (-12 + 24\mathrm{R})(\mathrm{Q} - 2\mathrm{Q}^3 + \mathrm{Q}^4) \qquad 27$$

Differentiating Equation 24 with respect to Q gives

$$\frac{\partial sh}{\partial Q} = (\mathrm{R} - 2\mathrm{R}^3 + \mathrm{R}^4)(1 - 6\mathrm{Q}^2 + 4\mathrm{Q}^3) \qquad 28$$

Differentiating Equation 24 with respect to Q twice gives

$$\frac{\partial^2 sh}{\partial Q^2} = (R - 2\mathrm{R}^3 + \mathrm{R}^4)(-12\mathrm{Q} + 12\mathrm{Q}^2) \qquad 29$$

Differentiating Equation 24 with respect to Q thrice gives

$$\frac{\partial^3 sh}{\partial Q^3} = (R - 2\mathrm{R}^3 + \mathrm{R}^4)(-12 + 24\mathrm{Q}) \qquad 30$$

Buth with respect to R and Q gives

$$\frac{\partial^2 sh}{\partial \mathrm{R}\,\partial \mathrm{Q}} = (1 - 6\mathrm{R}^2 + 4\mathrm{R}^3)(1 - 6\mathrm{Q}^2 + 4\mathrm{Q}^3) \qquad 31$$

Similarly with respect to R and twice Q gives

$$\frac{\partial^3 sh}{\partial \mathrm{R}\,\partial Q^2} = (1 - 6\mathrm{R}^2 + 4\mathrm{R}^3)(-12\mathrm{Q} + 12\mathrm{Q}^2) \qquad 32$$

$$\frac{\partial^3 sh}{\partial R^3} = (-12 + 24\mathrm{R})(\mathrm{Q} - 2\mathrm{Q}^3 + 4\mathrm{Q}^4) \qquad 28$$

$$\overline{Ct1} = \frac{\partial^3 sh}{\partial R^3} * \frac{\partial \mathrm{sh}}{\partial \mathrm{R}} \qquad 29$$

$$\overline{Ct2} = \frac{\partial^3 sh}{\partial \mathrm{R}\,\partial Q^2} * \frac{\partial \mathrm{sh}}{\partial \mathrm{R}} \qquad 30$$

$$\overline{\mathrm{Ct3}} = \frac{\partial^3 \mathrm{sh}}{\partial \mathrm{Q}^3} * \frac{\partial \mathrm{sh}}{\partial \mathrm{Q}} \qquad 31$$

$$\overline{Ct6} = \frac{\partial^2 sh}{\partial R^2} \qquad 32$$

Combining and multiplying from the expressions above gives,

$(-12 + 24\mathrm{R})(\mathrm{Q} - 2\mathrm{Q}^3 + \mathrm{Q}^4)x(1 - 6\mathrm{R}^2 + 4\mathrm{R}^3)(\mathrm{Q} - 2\mathrm{Q}^3 + \mathrm{Q}^4)$

Collecting the like terms together yields

$(-12 + 24\mathrm{R})(1 - 6\mathrm{R}^2 + 4\mathrm{R}^3)\ \mathrm{x}\ (\mathrm{Q} - 2\mathrm{Q}^3 + \mathrm{Q}^4)(\mathrm{Q} - 2\mathrm{Q}^3 + \mathrm{Q}^4)$

and multiplying out each bracket gives

$-12(1-6R^2+4R^3)+24R(1-6R^2+4R^3)$ x $Q(Q-2Q^3+Q^4)-2Q^3(Q-2Q^3+Q^4)$

$+\,Q^4(Q-2Q^3+Q^4)$

which finally changes to

$(-12+72R^2-48R^3+24R-144R^3+96R^4)x(Q^2-2Q^4+Q^5-2Q^4+4Q^6-2Q^7+Q^5-2Q^7+Q^8)$

Therefore $\frac{\partial^3 sh}{\partial R^3}*\frac{\partial sh}{\partial R}=(-12+24R+72R^2-192R^3+96R^4)$

x $(Q^2-4Q^4+2Q^5+4Q^6-4Q^7+Q^8)$ 33

But $ct_1=\iint\overline{Crt1}\,dRdQ=\iint\frac{\partial^3 sh}{\partial R^3}*\frac{\partial sh}{\partial R}\,dRdQ$

That implies that $cr_1=\int_0^1\int_0^1(-12+24R+72R^2-192R^3+96R^4)$ x

$(Q^2-4Q^4+2Q^5+4Q^6-4Q^7+Q^8)dRdQ$ 34

$=\ [[(\frac{-12R}{1}+\frac{24R^2}{2}+\frac{72R^3}{3}-\frac{192R^4}{4}+\frac{96R^5}{5})$ x $(\frac{Q^3}{3}-\frac{4Q^5}{5}+\frac{2Q^3}{6}+\frac{4Q^7}{7}-\frac{4Q^8}{8}+\frac{Q^9}{9})]^1]^1$

$=(-12+\frac{24}{2}+\frac{72}{3}-\frac{192}{4}+\frac{96}{5})$ x $(\frac{1}{3}-\frac{4}{5}+\frac{2}{6}+\frac{4}{7}-\frac{4}{8}+\frac{1}{9})$

Therefore $ct_1=(-4\frac{4}{5})$ x $(\frac{31}{630})=\frac{-124}{525}$

also,

From Equation 30 comes $(1-6R^2+4R^3)(-12Q+12Q^2)$ x $(1-6R^2+4R^3)(Q-2Q^3+Q^4)$

Bringing the similar terms together gives

$=(1-6R^2+4R^3)(1-6R^2+4R^3)$ x $(-12Q+12Q^2)(Q-2Q^3+Q^4)$

and multiplying out each bracket gives

$=1(1-6R^2+4R^3)-6R^2(1-6R^2+4R^3)+4R^3(1-6R^2+4R^3)$ x

$-12Q(Q-2Q^3+Q^4)+12Q^2(Q-2Q^3+Q^4)$

which finally changes to

$(1-6R^2+4R^3-6R^2+36R^4-24R^5+4R^3-24R^5+16R^6)x(-12Q^2+24Q^4-12Q^5+12Q^3-24Q^5+12Q^6)$

$=(1-12R^2+8R^3+36R^4-48R^5+16R^6)x(-12Q^2+12Q^3+24Q^4-36Q^5+12Q^6)$

Therefore $\frac{\partial^3 h}{\partial R\,\partial Q^2}*\frac{\partial h}{\partial R}=(1-12R^2+8R^3+36R^4-48R^5+16R^6)x$

$(-12Q^2+12Q^3+24Q^4-36Q^5+12Q^6)$ 35

But $ct_2=\iint\overline{Ct2}\,dRdQ=\iint\frac{\partial^3 h}{\partial R^3}*\frac{\partial h}{\partial R}\,dRdQ$

That implies that $crt_2=\int_0^1\int_0^1(1-12R^2+8R^3+36R^4-48R^5+16R^6)$ x

$(-12Q^2+12Q^3+24Q^4-36Q^5+12Q^6)dRdQ$ 36

$=[[(\frac{R}{1}+\frac{12R^3}{3}+\frac{8R^4}{4}+\frac{36R^5}{5}-\frac{48R^6}{6}+\frac{16R^7}{7})$ x $(\frac{-12Q^3}{3}-\frac{12Q^4}{4}+$

$\frac{24Q^5}{5}-\frac{36Q^6}{6}+\frac{12^7}{7})]^1]^1$

$=(\frac{1}{1}+\frac{12}{3}-\frac{8}{4}+\frac{36}{5}-\frac{48}{6}+\frac{16}{7})$ x $(\frac{-12}{3}+\frac{12}{4}+\frac{24}{5}+\frac{36}{6}-\frac{12}{7})$

Therefore $k_2=(\frac{17}{35})$ x $(\frac{-17}{35})=\frac{-289}{1225}$

also,

from Equation 31 $= (R - 2R^3 + R^4)(-12 + 24Q)\ x(R - 2R^3 + R^4)(1 - 6Q^2 + 4Q^3)$

Collecting the like terms together yields

$(R - 2R^3 + R^4)(R - 2R^3 + R^4)\ x\ (-12 + 24Q)(1 - 6Q^2 + 4Q^3)$

and multiplying out each bracket, gives

$= R(R - 2R^3 + R^4) - 2R^3(R - 2R^3 + R^4) + R^4(R - 2R^3 + R^4)x\ - 12(1 - 6Q^2 + 4Q^3)$
$+24Q(1 - 6Q^2 + 4Q^3)$

which finally changes to

$(R^2 - 2R^4 + R^5 - 2R^4 + 4R^6 - 2R^7 + R^5 - 2R^7 + R^8)x\ (-12 + 72Q^2 - 48Q^3 + 24Q - 144Q^3 + 96Q^4)$

$=(R^2 - 4R^4 + 2R^5 + 4R^6 - 4R^7 + R^8)x\ (-12 + 24Q + 72Q^2 - 192Q^3 + 96Q^4)$

Therefore $\frac{\partial^3 sh}{\partial Q^3} * \frac{\partial sh}{\partial Q} = (R^2 - 4R^4 + 2R^5 + 4R^6 - 4R^7 + R^8)x\ (-12 + 24Q + 72Q^2 - 192Q^3 + 96Q^4)$

But $crt_3 = \iint \overline{Crt3}dRdQ = \iint \frac{\partial^3 sh}{\partial R^3} * \frac{\partial sh}{\partial R} dRdQ$

That implies that

$ct_3 = \int_0^1 \int_0^1 (R^2 - 4R^4 + 2R^5 + 4R^6 - 4R^7 + R^8)$

$x\ (-12 + 24Q + 72Q^2 - 192Q^3 + 96Q^4)dRdQ$ 37

$= [[(\frac{R^3}{3} + \frac{4R^5}{5} + \frac{2R^6}{6} + \frac{4R^7}{7} - \frac{4R^8}{8} + \frac{R^9}{9})\ x\ (\frac{-12Q}{1} + \frac{24Q^2}{2} + \frac{72Q^3}{3}$

$+ \frac{192Q^4}{4} + \frac{96Q^5}{5})]^1]^1$

$=(\frac{1}{3} + \frac{4}{5} + \frac{2}{6} + \frac{4}{7} - \frac{4}{8} + \frac{1}{9})\ x\ (\frac{-12}{1} + \frac{24}{2} + \frac{72}{3} - \frac{192}{4} + \frac{96}{5})$

Therefore $k_3 = (\frac{31}{630})\ x\ (-4\frac{4}{35}) = \frac{-124}{525}$ also

$$(\frac{\partial sh}{\partial R})^2 = (1 - 6R^2 + 4R^3)(Q - 2Q^3 + Q^4)x(1 - 6R^2 + 4R^3)(Q - 2Q^3 + Q^4)$$

Collecting the like terms together yields

$$(1 - 6R^2 + 4R^3)(1 - 6R^2 + 4R^3)x(Q - 2Q^3 + Q^4)(Q - 2Q^3 + Q^4)$$

and multiplying out each bracket gives

$1(1 - 6R^2 + 4R^3) - 6R^2(1 - 6R^2 + 4R^3) + 4R^3(1 - 6R^2 + 4R^3)x\ Q(Q - 2Q^3 + Q^4) - 2Q^3(Q - 2Q^3 + Q^4) + Q^4(Q - 2Q^3 + Q^4)$

$= (1 - 6R^2 + 4R^3 - 6R^2 + 36R^4 - 24R^5 + 4R^3 - 24R^5 + 16R^6)x(Q^2 - 2Q^4 + Q^5 - 2Q^4 + 4Q^6 - 2Q^7 + Q^5 - 2Q^7 + Q^8)$

which finally changes to

$$(1 - 12R^2 + 8R^3 + 36R^4 - 48R^5 + 16R^6)x(Q^2 - 4Q^4 + 2Q^5 + 4Q^6 - 4Q^7 + Q^8)$$

Therefore$\left(\frac{\partial sh}{\partial R}\right)^2 = (1 - 12R^2 + 8R^3 + 36R^4 - 48R^5 + 16R^6)x$

$(Q^2 - 4Q^4 + 2Q^5 + 4Q^6 - 4Q^7 + Q^8)$

But $crt_6 = \iint \overline{Crt6}dRdQ = \iint (\frac{\partial sh}{\partial R})^2 dRdQ$

That implies that $crt_6 = \int_0^1 \int_0^1 (1 - 12R^2 + 8R^3 + 36R^4 - 48R^5 + 16R^6)\ x$

$(Q^2 - 4Q^4 + 2Q^5 + 4Q^6 - 4Q^7 + Q^8)dRdQ47$

$= [[(\frac{R}{1} - \frac{12R^3}{3} + \frac{8R^4}{4} + \frac{36R^5}{5} - \frac{48R^6}{6} + \frac{16R^7}{7})\ x\ (\frac{Q^3}{3} - \frac{4Q^5}{5} + \frac{2Q^6}{6}$

$+\frac{4Q^7}{7}-\frac{4Q^8}{8}+\frac{Q^9}{9})]^1]^1$

$=[[(\frac{1}{1}-\frac{12}{3}+\frac{8}{4}+\frac{36}{5}-\frac{48}{6}+\frac{16}{7})\ x\ (\frac{1}{3}-\frac{4}{5}+\frac{2}{6}+\frac{4}{7}-\frac{4}{8}+\frac{1}{9})]^1]^1$

Therefore $crt_6 = (\frac{17}{35})\ x\ (\frac{31}{630}) = \frac{527}{22050}$

Similarly from Polynomial rules, the shape function, sh for CCCC panel

is $(R^2-2R^3+R^4)(Q^2-2Q^3+Q^4)$ 38

Differential values for Clamped-Clamped-Clamped-Clamped shape

Also the various differential values for CCCC shape functions are as detail

$$sh = (R^2 - 2R^3 + R^4)(Q^2 - 2Q^3 + Q^4) \quad 39$$

Differentiating Equation 39 with respect to R gives

$$\frac{\partial sh}{\partial R} = (2R - 6R^2 + 4R^3)(Q^2 - 2Q^3 + Q^4) \quad 40$$

Differentiating Equation 39 with respect to R twice gives

$$\frac{\partial^2 sh}{\partial R^2} = (2 - 12R + 12R^2)(Q^2 - 2Q^3 + Q^4) \quad 41$$

Differentiating Equation 39 with respect to R thrice gives

$$\frac{\partial^3 sh}{\partial R^3} = (-12 + 24R)(Q^2 - 2Q^3 + Q^4) \quad 42$$

Differentiating Equation 39 with respect to Q gives

$$\frac{\partial sh}{\partial Q} = (R^2 - 2R^3 + R^4)(2Q - 6Q^2 + 4Q^3) \quad 43$$

Differentiating Equation 39 with respect to Q twice gives

$$\frac{\partial^2 sh}{\partial Q^2} = (R^2 - 2R^3 + R^4)(2 - 12Q + 12Q^2) \quad 44$$

Differentiating Equation 39 with respect to Q thrice gives

$$\frac{\partial^3 sh}{\partial Q^3} = (R^2 - 2R^3 + R^4)(-12 + 24Q) \quad 45$$

Differentiating Equation 39 with respect to Q thrice gives

$$\frac{\partial^2 sh}{\partial R\,\partial Q} = (2R - 6R^2 + 4R^3)(2Q - 6Q^2 + 4Q^3)56$$

$$\frac{\partial^3 sh}{\partial R\,\partial Q^2} = (2R - 6R^2 + 4R^3)(2 - 12Q + 12Q^2) \quad 46$$

$$\overline{Ct1} = \frac{\partial^3 sh}{\partial R^3} * \frac{\partial sh}{\partial R} \quad 47$$

$\overline{Ct2} = \frac{\partial^3 sh}{\partial R\,\partial Q^2} * \frac{\partial sh}{\partial R}$ 48

$\overline{Ct3} = \frac{\partial^{3s} h}{\partial Q^3} * \frac{\partial sh}{\partial Q}$ 49

$\overline{Crt6} = \frac{\partial^2 sh}{\partial R^2}$ or $(\frac{\partial sh}{\partial R})^2$ 61

From the expressions above,

$$\frac{\partial^3 sh}{\partial R^3} * \frac{\partial sh}{\partial R} = (-12 + 24R)(Q^2 - 2Q^3 + Q^4) \text{ x } (2R - 6R^2 + 4R^3)(Q^2 - 2Q^3 + Q^4)$$

Collecting the like terms together yields

$$(-12 + 24R)(2R - 6R^2 + 4R^3) \text{ x } (Q^2 - 2Q^3 + Q^4)(Q^2 - 2Q^3 + Q^4)$$

and multiplying out each bracket gives

$$-12(2R - 6R^2 + 4R^3) + 24R(2R - 6R^2 + 4R^3) \text{ x } Q^2(Q^2 - 2Q^3 + Q^4)$$

$$-2Q^3(Q^2 - 2Q^3 + Q^4) + Q^4(Q^2 - 2Q^3 + Q^4)$$

$$= (-24R + 72R^2 - 48R^3 + 48R^2 - 144R^3 + 96R^4)$$

$$x(Q^4 - 2Q^5 + Q^6 - 2Q^5 + 4Q^6 - 2Q^7 + Q^6 - 2Q^7 + Q^8)$$

which finally changes to

$$(-24R + 120R^2 - 192R^3 + 96R^4)x(Q^4 - 4Q^5 + 6Q^6 - 4Q^7 + Q^8)$$

Therefore $\frac{\partial^3 sh}{\partial R^3} * \frac{\partial sh}{\partial R} = (-24R + 120R^2 - 192R^3 + 96R^4)x(Q^4 - 4Q^5 + 6Q^6 - 4Q^7 + Q^8)$

But $crt_1 = \iint \overline{Crt1}\,dRdQ = \iint \frac{\partial^3 h}{\partial R^3} * \frac{\partial h}{\partial R}\,dRdQ$

That implies that

$ct_1 = \int_0^1 \int_0^1 (-24R + 120R^2 - 192R^3 + 96R^4) \text{ x}(Q^4 - 4Q^5 + 6Q^6 - 4Q^7 + Q^8)dRdQ$ 50

$= [[(\frac{-24R^2}{2} + \frac{120R^3}{3} - \frac{192R^4}{4} + \frac{96R^5}{5}) \text{ x } (\frac{Q^5}{5} - \frac{4Q^6}{6} + \frac{6Q^7}{7} - \frac{4Q^8}{8} + \frac{Q^9}{9})]^1]^1$

$= [[(\frac{-24}{2} + \frac{120}{3} - \frac{192}{4} + \frac{96}{5}) \text{ x } (\frac{1}{5} - \frac{4}{6} + \frac{6}{7} - \frac{4}{8} + \frac{1}{9})$

Therefore

$ct_1 = (-\frac{4}{5}) \text{ x } (\frac{1}{630}) = \frac{-4}{3150}$also**,**

$$\frac{\partial^3 sh}{\partial R\,\partial Q^2} * \frac{\partial sh}{\partial R} = (2R - 6R^2 + 4R^3)(2 - 12Q + 12Q^2)x(2R - 6R^2 + 4R^3)(Q^2 - 2Q^3 + Q^4)$$

Collecting the like terms together yields

$$= (2R - 6R^2 + 4R^3)(2R - 6R^2 + 4R^3) \text{ x } (2 - 12Q + 12Q^2)(Q^2 - 2Q^3 + Q^4)$$

and multiplying out each bracket gives

$$2\text{Rx}\, 2(\text{Q}^2 - 2\text{Q}^3 + \text{Q}^4) - 12\text{Q}(\text{Q}^2 - 2\text{Q}^3 + \text{Q}^4) + 12\text{Q}^2(\text{Q}^2 - 2\text{Q}^3 + \text{Q}^4)$$

$= (4\text{R}^2 - 12\text{R}^3 + 8\text{R}^4 - 12\text{R}^3 + 36\text{R}^4 - 24\text{R}^5 + 8\text{R}^4 - 24\text{R}^5 + 16\text{R}^6)$

$\text{x}\ (2\text{Q}^2 - 4\text{Q}^3 2\text{Q}^4 - 12\text{Q}^3 + 12\text{Q}^4 - 12\text{Q}^5 + 12\text{Q}^4 + 24\text{Q}^5 + 12\text{Q}^6)$

which finally changes to

$(4\text{R}^2 - 24\text{R}^3 + 52\text{R}^4 - 48\text{R}^5 + 16\text{R}^6)x(2\text{Q}^2 - 16\text{Q}^3 + 38\text{Q}^4 - 36\text{Q}^5 + 12\text{Q}^6)$`

Therefore

$$\frac{\partial^3 sh}{\partial R^3} * \frac{\partial \text{sh}}{\partial \text{R}} = (4\text{R}^2 - 24\text{R}^3 + 52\text{R}^4 - 48\text{R}^5 + 16\text{R}^6)x(2\text{Q}^2 - 16\text{Q}^3 + 38\text{Q}^4 - 36\text{Q}^5 + 12\text{Q}^6)$$

But $ct_2 = \iint \overline{Crt2}\text{dRdQ} = \iint \frac{\partial^3 h}{\partial \text{R}\, \partial Q^2} * \frac{\partial \text{h}}{\partial \text{R}}\text{dRdQ}$

That implies that

$ct_2 = \int_0^1 \int_0^1 (4\text{R}^2 - 24\text{R}^3 + 52\text{R}^4 - 48\text{R}^5 + 16\text{R}^6)\ \text{x}\ (2\text{Q}^2 - 16\text{Q}^3 + 38\text{Q}^4 - 36\text{Q}^5 + 12\text{Q}^6)$
dRdQ 51

$= [[(\frac{4\text{R}^3}{3} + \frac{24\text{R}^4}{4} + \frac{52\text{R}^5}{5} - \frac{48\text{R}^7}{6} + \frac{16\text{R}^8}{7})\ \text{x}\ (\frac{2\text{Q}^3}{3} - \frac{16\text{Q}^4}{4} + \frac{38\text{Q}^5}{5} \frac{36\text{Q}^6}{6} + \frac{12^7}{7})]^1]^1$

$= (\frac{4}{3} - -\frac{24}{4} + \frac{52}{5} - \frac{48}{6} + \frac{16}{7})\ \text{x}\ (\frac{2}{3} - \frac{16}{4} + \frac{38}{5} - \frac{36}{6} + \frac{12}{7})$

Therefore $crt_2 = (\frac{2}{105})\ \text{x}\ (\frac{-2}{105}) = \frac{-4}{11025}$
also,

$$\frac{\partial^3 sh}{\partial Q^3} * \frac{\partial \text{sh}}{\partial \text{Q}} = (\text{R}^2 - 2\text{R}^3 + \text{R}^4)(-12 + 24\text{Q})\text{x}(\text{R}^2 - 2\text{R}^3 + \text{R}^4)(2\text{Q} - 6\text{Q}^2 + 4\text{Q}^3)$$

Collecting the like terms together yields

$$(\text{R}^2 - 2\text{R}^3 + \text{R}^4)(\text{R}^2 - 2\text{R}^3 + \text{R}^4)\ \text{x}\ (-12 + 24\text{Q})(2\text{Q} - 6\text{Q}^2 + 4\text{Q}^3)$$

and multiplying out each bracket gives

$\text{R}^2(\text{R}^2 - 2\text{R}^3 + \text{R}^4) - 2\text{R}^3(\text{R}^2 - 2\text{R}^3 + \text{R}^4) + \text{R}^4(\text{R}^2 - 2\text{R}^3 + \text{R}^4)$

$$\text{x} - 12(2\text{Q} - 6\text{Q}^2 + 4\text{Q}^3) + 24\text{Q}(2\text{Q} - 6\text{Q}^2 + 4\text{Q}^3)$$

$= (\text{R}^4 - 2\text{R}^5 + \text{R}^6 - 2\text{R}^5 + 4\text{R}^6 - 2\text{R}^7 + \text{R}^6 - 2\text{R}^7 + \text{R}^8)$

$\text{x}\ (-24\text{Q} + 72\text{Q}^2 - 48\text{Q}^3 + 48\text{Q}^2 - 144\text{Q}^3 + 96\text{Q}^4)$

which finally changes to

$(\text{R}^4 - 4\text{R}^5 + 6\text{R}^6 - 4\text{R}^7 + \text{R}^8)\ \text{x}\ (-24\text{Q} + 120\text{Q}^2 - 192\text{Q}^3 + 96\text{Q}^4)$

Therefore

$$\frac{\partial^3 sh}{\partial Q^3} * \frac{\partial \text{sh}}{\partial \text{Q}} = (\text{R}^4 - 4\text{R}^5 + 6\text{R}^6 - 4\text{R}^7 + \text{R}^8)\ \text{x}\ (-24\text{Q} + 120\text{Q}^2 - 192\text{Q}^3 + 96\text{Q}^4)$$

But $crt_3 = \iint \overline{Crt3}dRdQ = \iint \frac{\partial^3 sh}{\partial Q^3} * \frac{\partial sh}{\partial Q}dRdQ$

That implies that

$ct_3 = \int_0^1 \int_0^1 (R^4 - 4R^5 + 6R^6 - 4R^7 + R^8) \text{ x } (-24Q + 120Q^2 - 192Q^3 + 96Q^4)dRdQ$ 52

$= [[(\frac{R^5}{5} - \frac{4R^6}{6} + \frac{6R^7}{7} - \frac{4R^8}{8} + \frac{R^9}{9}) \text{ x } (\frac{-24Q^2}{2} + \frac{120Q^3}{3} - \frac{192Q^4}{4} + \frac{96Q^5}{5})]^1]^1$

$= (\frac{1}{5} - \frac{4}{6} + \frac{6}{7} - \frac{4}{8} + \frac{1}{9}) \text{ x } (-\frac{24}{2} + \frac{120}{3} - \frac{192}{4} + \frac{96}{5})$

Therefore $crt_3 = (\frac{1}{630}) \text{ x } (-\frac{4}{5}) = \frac{-4}{3150} = \frac{-2}{1575}$

Also

$(\frac{\partial sh}{\partial R})^2 = (2R - 6R^2 + 4R^3)(Q^2 - 2Q^3 + Q^4) \text{x } (2R - 6R^2 + 4R^3)(Q^2 - 2Q^3 + Q^4)$

Collecting the like terms together yields

$(2R - 6R^2 + 4R^3)(2R - 6R^2 + 4R^3) \text{ x } (Q^2 - 2Q^3 + Q^4)(Q^2 - 2Q^3 + Q^4)$

and multiplying out each bracket gives

$2R(2R - 6R^2 + 4R^3) - 6R^2(2R - 6R^2 + 4R^3) + 4R^3(2R - 6R^2 + 4R^3)$

$\text{x } Q^2(Q^2 - 2Q^3 + Q^4) - 2Q^3(Q^2 - 2Q^3 + Q^4) + Q^4(Q^2 - 2Q^3 + Q^4)$

$= (4R^2 - 12R^3 + 8R^4 - 12R^3 + 36R^4 - 24R^5 + 8R^4 - 24R^5 + 16R^6)$

$\text{x}(Q^4 - 2Q^5 + Q^6 - 2Q^5 + 4Q^6 - 2Q^7 - Q^6 - 2Q^7 + Q^8)$

which finally changes to

$(4R^2 - 24R^3 + 52R^4 - 48R^5 + 16R^6) \text{ x } (Q^4 - 4Q^5 + 6Q^6 - 4Q^7 + Q^8)$

Therefore

$(\frac{\partial sh}{\partial R})^2 = (4R^2 - 24R^3 + 52R^4 - 48R^5 + 16R^6) \text{ x } (Q^4 - 4Q^5 + 6Q^6 - 4Q^7 + Q^8)$

But $crt_6 = \iint \overline{Crt6}dRdQ = \iint (\frac{\partial sh}{\partial R})^2 dRdQ$

That implies that

$ct_6 = \int_0^1 \int_0^1 (4R^2 - 24R^3 + 52R^4 - 48R^5 + 16R^6) \text{ x}(Q^4 - 4Q^5 + 6Q^6 - 4Q^7 + Q^8)dRdQ$ 53

$= [[(\frac{4R^3}{3} - \frac{24R^4}{4} + \frac{52R^5}{5} - \frac{48R^6}{6} + \frac{16R^7}{7}) \text{ x } (\frac{Q^5}{5} - \frac{4Q^6}{6} + \frac{6Q^7}{7} - \frac{4Q^8}{8} + \frac{Q^9}{9})]^1]^1$

$= [[(\frac{4}{3} - \frac{24}{4} + \frac{52}{5} - \frac{48}{6} + \frac{16}{7}) \text{ x } (\frac{1}{5} - \frac{4}{6} + \frac{6}{7} - \frac{4}{8} + \frac{1}{9})]^1]^1$

Therefore $crt_6 = (\frac{2}{105}) \text{ x } (\frac{1}{630}) = \frac{2}{66150}$

II. Results

When the stiffness coefficients were substituted into the critical buckling load equations the followings were obtained

Table 1: Shape functions and stiffness coefficients

Shape Functions, sh	Stiffness Coefficients			
	ct_1	ct_2	ct_3	ct_6
SSSS sh = (R-2R3+R4) x (Q-2Q3+Q4)	= 0.236219	= 0.23591	= 0.236219	= 0.02390
CCCC sh =$(R^2 - 2R^3 + R^4)$ x $(Q^2 - 2Q^3 + Q^4)$	= 0.00127	= 0.00036	= 0.00127	= 0.00003

Result for Simple-Simple-Simple-Simple plate

The non- dimensional buckling load parameters for SSSS plate are presented on Table 2.1 and 2.2 and Figure.1.

Table 2.1 Non dimensional buckling load parameters for SSSS plate for aspect ratio of b/a

b/a		2	1.9	1.8	1.7	1.6
B		15.4371	16.111	16.9186	17.8983	19.1036
$B_{crt}(\frac{D}{a^2})$	Previous	15.43632	16.11018	16.91777	17.89753	19.10282
	Present	15.43632	16.11018	16.91777	17.89753	19.10282

Table 2.2

b/a		1.5	1.4	1.3	1.2	1.1	1
B		20.610	22.529	25.026	28.359	32.950	39.505
$B_{crt}(\frac{D}{a^2})$	Previous	20.6103	22.5282	25.0256	28.357	32.9491	39.510
	Present	20.6101	22.5281	25.0250	28.358	32.949	39.508

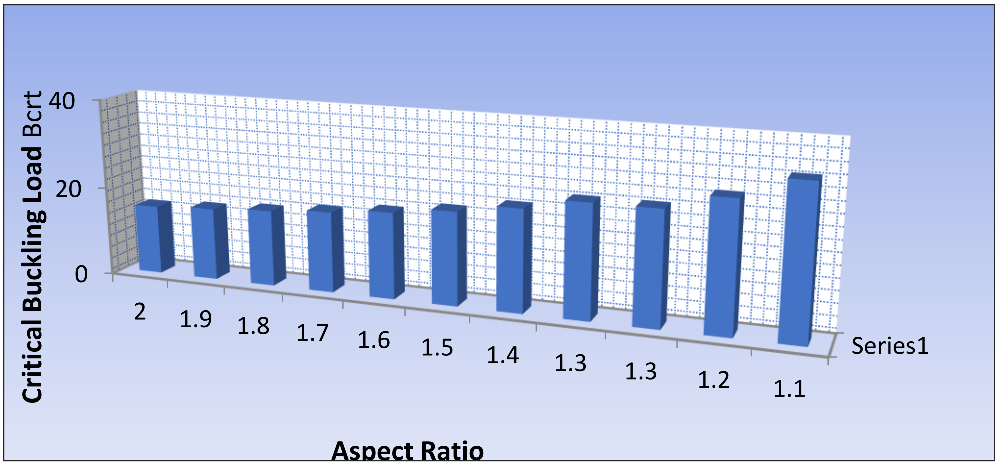

Figure 1.2: Buckling load against aspect ratio (SSSS) Present

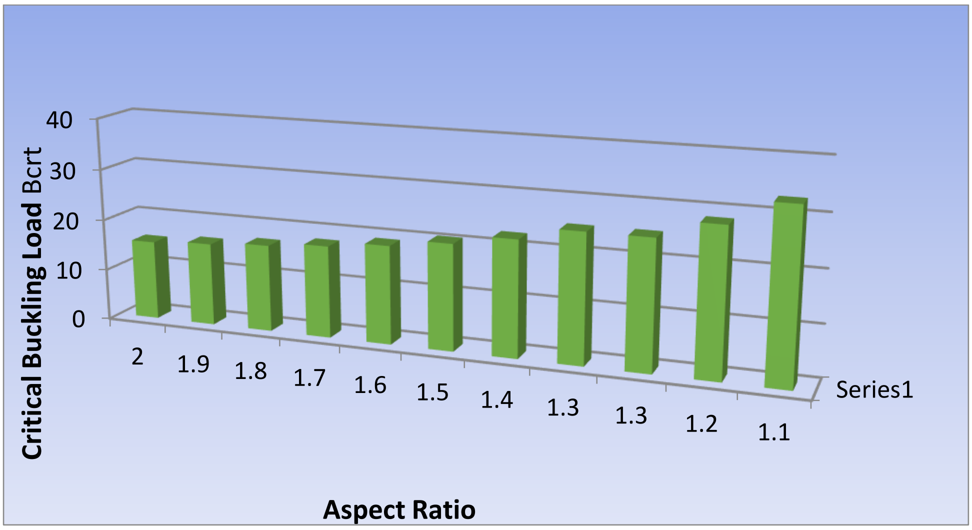

Figure 1.2: Buckling load against aspect ratio (SSSS) Previous

Table 2.2: B-values from present study compared with previous worksfor SSSS rectangular plate buckling.

Aspect Ratios	B_{crt}-Values from Present Study (1)	B_{crt}-Valuesfrom Previous 2i	B_{crt}-Values from Previous 2ii	Percentage Difference Between (1) and (2)
1	39.508	39.508	39.488	0
1.1	32.948	32.949	32.932	-0.00091
1.2	28.3589	28.359	28.344	-0.00176
1.3	25.025	25.0256	25.011	-0.0024
1.4	22.528	22.529	22.520	-0.00311
1.5	20.610	20.6102	20.597	-0.0034
1.6	19.103	19.1036	19.090	-0.00419
1.7	17.8978	17.8983	17.890	-0.00447
1.8	16.9178	16.9186	16.910	-0.00473

1.9	16.110	16.111	16.099	-0.0050
2	15.436	15.437	15.425	-0.00518

Result for clamped-clamped-clamped-clamped plate

Similarly, the non-dimensional buckling load values for CCCC plates are presented on Table 3.1 and 3.2 and Figure 2

Table 3.1: Non dimensional buckling load parameters for CCCC plate for aspect ratio of b/a

b/a		2	1.9	1.8	1.7	1.6
B		50.979	52.229	53.773	55.706	58.167
$B_{crt}(\frac{D}{a^2})$	Previous	50.979	52.229	53.773	55.706	58.167
	Present	50.9792	52.229	53.773	55.7064	58.167

Table 3.2

b/a		1.5	1.4	1.3	1.2	1.1	1
B		61.362	65.597	71.356	79.415	91.082	108.666
$B_{crt}(\frac{D}{a^2})$	Previous	61.362	65.597	71.356	79.415	91.082	108.667
	Present	61.362	65.597	71.356	79.4158	91.082	108.666

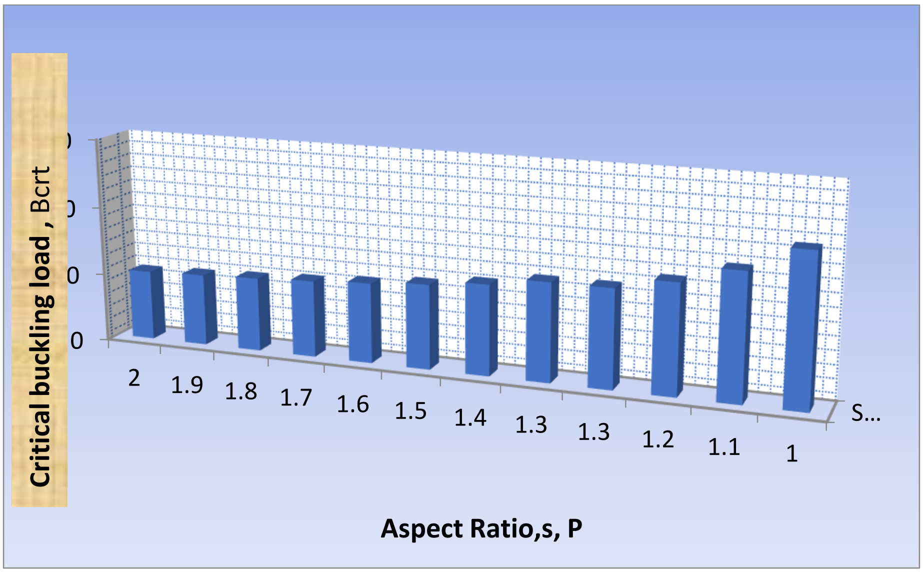

Figure iii: buckling load, B_{ct} against aspect ratio, p = b/a for CCCC (Present)

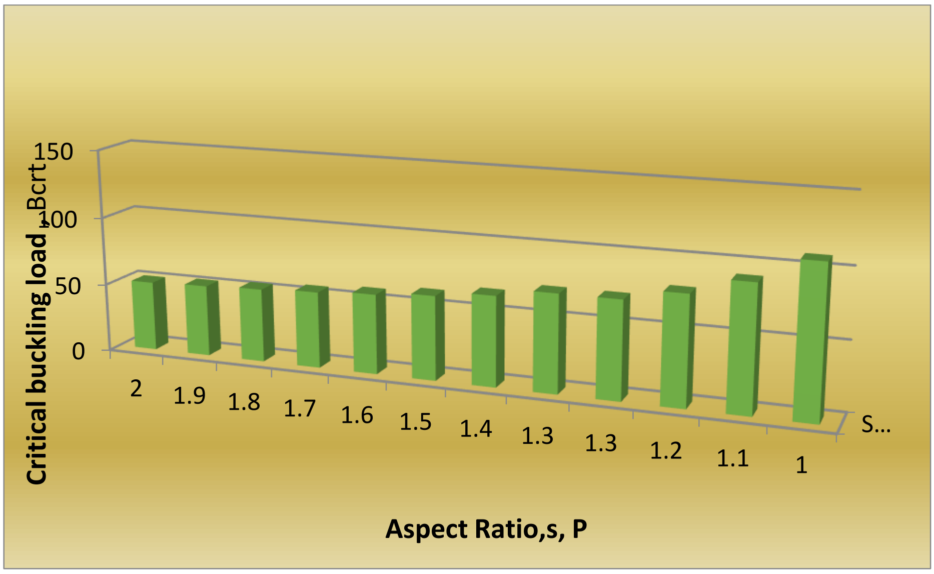

Figure iv : buckling load, B_{ct} against aspect ratio, p = b/afor CCCC (Previous)

Table 3b: B-values from present study compared with previous works for different aspectratio for CCCC rectangular plate buckling.

Aspect Ratios	B-Values from Present Study (3)	B-Values from Previous 4i	B-Values from Previous 4ii	Percentage Difference Between (3) and (4)
1	108.6667	108.667	108.654	-0.00028
1.1	91.0823	91.082	91.068	0.000329
1.2	79.41538	79.415	79.402	0.000478
1.3	71.3566	71.3565	71.3425	0.00014
1.4	65.598	65.5979	65.5829	0.000152
1.5	61.3621	61.3621	61.3501	0
1.6	58.1679	58.167	58.151	0.001547
1.7	55.7064	55.706	55.694	0.000718
1.8	53.7734	53.773	53.761	0.000744
1.9	52.2299	52.229	52.212	0.001723
2	50.9792	50.979	50.965	0.000392

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