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A note on L^1–Convergence of Rees- Stanojević Sum

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Abstract:- We obtain a necessary and sufficient condition for L^1–convergence of Rees-Stanojević [5] modified cosine sum and generalize a result of Garrett and Stanojević [2] and a classical result [1, p. 204].

Keywords:- L^1-convergence, modified cosine sum, class R.

I. INTRODUCTION

Consider the cosine series

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx . \tag{1.1}$$

Let $S_n(x)$ denote the partial sum of (1.1) and $f(x) = \lim_{n\to\infty} S_n(x)$.
Rees and Stanojević [5] introduced a modified cosine sum

$$f_n(x) = \frac{1}{2}\sum_{k=0}^{n} \Delta a_k + \sum_{k=1}^{n} \sum_{j=k}^{n} \Delta a_j \cos kx, \tag{1.2}$$

and obtained a necessary and sufficient condition for its integrability.
A null sequence $\{a_k\}$ is said to be quasi-convex if

$$\sum_{k=1}^{\infty} (k+1) |\Delta^2 a_k| < \infty .$$

Regarding the convergence of $f_n(x)$ in L-metric, Garrett and Stanojević [2] proved the following theorem :

Theorem A. If $\{a_k\}$ is a null quasi-convex sequence, then $f_n(x)$ converges to $f(x)$ in L-metric.
The following results are known :

Theorem B [1, p. 202]**.** If $\{a_k\}$ is a quasi-convex null sequence, then (1.1) is a Fourier series of its point-wise limit.

Theorem C [1, p. 204]. If $a_k \downarrow 0$ and $\{a_k\}$ is convex or even quasi-convex, then for L^1-convergence of (1.1) it is necessary and sufficient that $\lim_{n\to\infty} a_n \log n = 0$.

Kano [4, Theorem E] generalized Theorem B by proving the result :

Theorem D. If $a_k \to 0$ as $k \to \infty$ and

$$\sum_{k=1}^{\infty} k^2 \left|\Delta^2\left(\frac{a_k}{k}\right)\right| < \infty , \tag{1.3}$$

then (1.1) is a Fourier series.

The aim of this paper is to generalize theorem A by obtaining a necessary and sufficient condition for L^1-convergence of $f_n(x)$ under (1.3).Our corollary is the extension of Theorem C .

II. LEMMA

For the proof of our result, we require the following lemma :

Lemma 1[3]. Let $D_n(x)$, $\bar{D}_n(x)$ and $K_n(x)$ denote Dirichlet, conjugate Dirichlet and Fejér kernels respectively, then

$$D_n(x) = K_n(x) + \frac{1}{n+1} \bar{D}'_n(x) .$$

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III. RESULTS

We prove the following result :

Theorem 1. Let $\{a_k\}$ satisfy (1.3). Then

(i) $\lim_{n\to\infty} f_n(x) = f(x)$ for $x \varepsilon (0,\pi]$ and $f \varepsilon L(0,\pi]$.

(ii) $\| f_n - f \| = o(1), \quad n \to\infty$.

Proof. (i) We have

$$f_n(x) = \frac{1}{2}\sum_{k=0}^{n}\Delta a_k + \sum_{k=1}^{n}\ \sum_{j=k}^{n}\Delta a_j \cos kx$$

$$= \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx - a_{n+1} D_n(x) .$$

Since $\{a_k\}$ is null sequence, $\lim_{n\to\infty} f_n(x) = f(x)$ for $x \varepsilon (0,\pi]$, and by Theorem D we have $f \varepsilon L(0,\pi]$.

(ii) Application of Lemma 1 yields

$$f(x) - f_n(x) = \sum_{k=n+1}^{\infty} a_k \cos kx + a_{n+1} D_n(x)$$

$$= \sum_{k=n+1}^{\infty} a_k \cos kx + \frac{a_{n+1}}{n+1}\bar{D}'_n(x) + a_{n+1} K_n(x)$$

$$= \lim_{m\to\infty} \frac{d}{dx}\left(\sum_{k=n+1}^{m}\frac{a_k}{k}\sin kx\right) + \frac{a_{n+1}}{n+1}\bar{D}'_n(x) + a_{n+1} K_n(x).$$

Using Abel's transformation twice, we have

$$f(x) - f_n(x) = \lim_{m\to\infty}\left[\sum_{k=n+1}^{m-1}\Delta\left(\frac{a_k}{k}\right)\bar{D}'_k(x) + \frac{a_m}{m}\bar{D}'_m(x) - \frac{a_{n+1}}{n+1}\bar{D}'_n(x)\right] + \frac{a_{n+1}}{n+1}\bar{D}'_n(x) + a_{n+1} K_n(x)$$

$$= \lim_{m\to\infty}\left[\sum_{k=n+1}^{m-2}(k+1)\Delta^2\left(\frac{a_k}{k}\right)\bar{K}'_k(x) + m\Delta\left(\frac{a_{m-1}}{m-1}\right)\bar{K}'_{m-1}(x) - (n+1)\,\Delta\left(\frac{a_{n+1}}{n+1}\right)\bar{K}'_n(x) + \frac{a_m}{m}\bar{D}'_n(x) + a_{n+1} K_n(x)\right]$$

$$= \sum_{k=n+1}^{\infty}(k+1)\,\Delta^2\left(\frac{a_k}{k}\right)\bar{K}'_k(x) - (n+1)\,\Delta\left(\frac{a_{n+1}}{n+1}\right)\bar{K}'_n(x) + a_{n+1} K_n(x) ,$$

where $\bar{K}_k(x)$ denotes the conjugate Fejér kernel. Thus

$$\int_{-\pi}^{\pi}|f(x) - f_n(x)|\,dx \le \sum_{k=n+1}^{\infty}(k+1)\left|\Delta^2\left(\frac{a_k}{k}\right)\right|\int_{-\pi}^{\pi}|\bar{K}'_k(x)|\,dx + (n+1)\left|\Delta\left(\frac{a_{n+1}}{n+1}\right)\right|\int_{-\pi}^{\pi}|\bar{K}'_n(x)|\,dx + \pi\, a_{n+1} ,$$

since $\int_{-\pi}^{\pi}|K_n(x)|\,dx = \pi$.

But, by Zygmund's Theorem [1, p. 458], we have

$$\int_{-\pi}^{\pi}|\bar{K}'_k(x)|\,dx = O(k).$$

Also,

$$\left|\Delta\left(\frac{a_{n+1}}{n+1}\right)\right| = \left|\sum_{k=n+1}^{\infty}\Delta^2\left(\frac{a_k}{k}\right)\right|$$

$$\leq \sum_{k=n+1}^{\infty}\frac{k^2}{k^2}\left|\Delta^2\left(\frac{a_k}{k}\right)\right|$$

$$\leq \frac{1}{(n+1)^2}\sum_{k=n+1}^{\infty}k^2\left|\Delta^2\left(\frac{a_k}{k}\right)\right|$$

$$= o\left(\frac{1}{(n+1)^2}\right).$$

Thus, it follows that

$$\int_{-\pi}^{\pi}|f(x)-f_n(x)|\,dx = O\left(\sum_{k=n+1}^{\infty}(k+1)^2\left|\Delta^2\left(\frac{a_k}{k}\right)\right|\right) + o(1) + o(1)$$

$$= o(1).$$

IV. CORROLARY

(i) Let $\{a_k\}$ satisfy (1.3), then $\| f(x) - S_n(x)\| = o(1)$, $n \to \infty$, if and only if $|a_{n+1}| \log n = o(1)$, $n\to\infty$.

Proof. We have

$$\int_{-\pi}^{\pi}|f(x)-S_n(x)|\,dx \leq \int_{-\pi}^{\pi}|f(x)-f_n(x)|\,dx + \int_{-\pi}^{\pi}|f_n(x)-S_n(x)|\,dx$$

$$= \int_{-\pi}^{\pi}|f(x)-f_n(x)|\,dx + \int_{-\pi}^{\pi}|a_{n+1}D_n(x)|\,dx,$$

and

$$\int_{-\pi}^{\pi}|a_{n+1}D_n(x)|\,dx = \int_{-\pi}^{\pi}|f_n(x)-S_n(x)|\,dx$$

$$\leq \int_{-\pi}^{\pi}|f(x)-S_n(x)|\,dx + \int_{-\pi}^{\pi}|f(x)-f_n(x)|\,dx.$$

Since,

$\lim_{n\to\infty}\int_{-\pi}^{\pi}|f(x)-f_n(x)|\,dx = 0$ by our Theorem 1 and $\int_{-\pi}^{\pi}|D_n(x)|\,dx$ behaves like log n for large n, the conclusion follows.

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