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On L^1-Convergence of Cosine Sums

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Abstract:- A necessary and sufficient condition for L^1-convergence of a modified trigonometric sum has been obtained which generalizes a result of Kumari and Ram[5]. A result of Fomin [2]follows as a corollary of our result.

Keywords:- Modified cosine sum, class F_p

I. INTRODUCTION

Let $S_n(x)$ denote the partial sum of the cosine series

$$(1.1) \qquad (a_0/2) + \sum_{k=1}^{\infty} a_k \cos kx ,$$

and let $f(x) = \lim_{n\to\infty} S_n(x)$, if it exists. The convergence of the series (1.1) in the metric space L, has been studied by various authors under different conditions on the coefficients a_k .

Theorem A [8]. If $\{a_k\}$ is convex ($\Delta^2 a_k \geq 0$, for every k) null sequence, then the cosine series (1.1) is the Fourier series of its sum f, and

$$\|S_n(x) - f(x)\| = o(1) \text{ , if and only if } a_n \log n = o(1),\ n \to \infty .$$

Kolmogorov extended the definition of convex null sequence. A sequence $\{a_k\}$ is said to be quasi-convex if $a_k = o(1)$, $k \to \infty$, and the series $\sum_{k=1}^{\infty} k\,|\Delta^2 a_k| < \infty$.

Theorem B[4]. If $\{a_k\}$ is a quasi-convex null sequence, then the cosine series (1.1) is the Fourier series of its sum f, and

$$\|S_n(x) - f(x)\| = o(1) \text{ , if and only if } a_n \log n = o(1),\ n\to\infty .$$

Sidon generalized the concept of convex (quasi-convex) null sequence. We say a sequence $\{a_k\}$ belongs to the class S[6], if there exists a monotonically decreasing sequence $\{A_k\}$ such that $\Sigma A_k < \infty$, and $|\Delta a_k| \leq A_k$, for every k.

A quasi-convex null sequence satisfies the class S if we take $A_n = \sum_{m=n}^{\infty} |\Delta^2 a_m|$.

Theorem C[7]. Let $\{a_k\}$ belong to the class S. Then the cosine series (1.1) is the Fourier series of its sum f and

$$\|S_n(x) - f(x)| = o(1), \text{ if and only if } a_n \log n = o(1),\ n\to\infty .$$

Fomin[2] generalized Teljakovskiĭ's Theorem by introducing a new class F_p . A null sequence $\{a_k\}$ belongs to the class F_p, if for some $1 < p \leq 2$, we have

$$\sum_{k=1}^{n}\left(\sum_{k=n}^{\infty}|\Delta a_k|^p \Big/ n\right)^{\frac{1}{p}} < \infty .$$

and proved that $F_p \subset BV$, where BV denotes the class of bounded variation and

if $\{a_k\}\ \varepsilon\ F_p$, then $f\ \varepsilon\ L^1(o, \pi)$ and $\sum_{k=1}^{\infty} k^{p-1}|\Delta a_k|^p < \infty$.

Theorem D [2]. Let $\{a_k\}$ satisfy the conditions of F_p ,then the cosine series (1.1) is the Fourier series of its sum f and

$$\|S_n(x) - f(x)\| = o(1) \text{ if and only if } a_n \log n = o(1),\ n\to\infty .$$

Kumari and Ram [5]introduced a new modified trigonometric cosine sum

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(1.2) $$f_n(x) = (a_0/2) + \sum_{k=1}^{n} \sum_{j=k}^{n} \Delta(a_j/j)\, k \cos kx ,$$

and proved

Theorem E[5]. Let the sequence $\{a_k\}$ satisfy the conditions of the class S.
If $a_n \log n = o(1), n\to\infty$, then $\|f(x) - f_n(x)\| = o(1), n\to\infty$.

In this paper, we generalize Theorem E by obtaining a necessary and sufficient condition for the convergence of the limit of (1.2) in the metric space L by taking $\{a_k\}$ in the class F_p. We also deduce Fomin's Theorem D as a corollary from our result.

II. LEMMA

We require the following Lemma for the proof of our result:

Lemma 1[3]. If $D_n(x)$, $\bar{D}_n(x)$ and $F_n(x)$ denote Dirichlet, conjugate Dirichlet and Fejér's kernel respectively, then $F_n(x) = D_n(x) - (1/n+1)\, \bar{D}'_n(x)$.

III. MAIN RESULT

Theorem 1. Let $\{a_k\}$ belong to the class F_p. Then
$\|f(x) - f_n(x)\| = o(1)$ if and only if $a_n \log n = o(1), n\to\infty$.

Proof. Application of Lemma 1 and summation by parts give

(3.1) $$\begin{aligned} f_n(x) &= \frac{a_0}{2} + \sum_{k=1}^{n} \sum_{j=k}^{n} \Delta\left(\frac{a_j}{j}\right) k \cos kx \\ &= S_n(x) - \left(\frac{a_{n+1}}{n+1}\right) \bar{D}'_n(x) \\ &= S_n(x) - a_{n+1} D_n(x) + a_{n+1} F_n(x) \\ &= \sum_{k=0}^{n} \Delta a_k D_k(x) + a_{n+1} F_n(x) \end{aligned}$$

Since $D_n(x) = O(1/x^2)$ and $a_n \to 0$ as $n\to\infty$, $\lim_{n\to\infty} f_n(x)$ exists and $f(x) = \lim_{n\to\infty} f_n(x)$. Now,

(3.2) $$\int_0^{\pi} |f_n(x) - f(x)|\, dx = \int_0^{\frac{1}{n}} |f_n(x) - f(x)|\, dx + \int_{\frac{1}{n}}^{\pi} |f_n(x) - f(x)|\, dx.$$

For the first integral of right hand side of (3.2),we have

$$\int_0^{\frac{1}{n}} |f_n(x) - f(x)|\, dx \le \int_0^{\frac{1}{n}} |\sigma_n(x) - f(x)| dx + \int_0^{\frac{1}{n}} |f_n(x) - \sigma_n(x)| dx,$$

where $\sigma_n(x)$ is the Fejér sum of $S_n(x)$, and

$$\int_0^{\frac{1}{n}} |\sigma_n(x) - f(x)| dx = O(\|\sigma_n(x) - f(x)\|),\ n\to\infty .$$

From (3.1), we get

$$f_n(x) - \sigma_n(x) = a_{n+1} F_n(x) + \left(\frac{1}{n+1}\right) \sum_{k=1}^{n} k \Delta a_k D_k(x) - \left(\frac{1}{n+1}\right) \sum_{k=1}^{n} a_k D_k(x)$$

Therefore,

$$\int_0^{\frac{1}{n}} |f_n(x) - \sigma_n(x)|\, dx \le (\pi/2)\, a_{n+1} + \left(\frac{1}{n+1}\right) \sum_{k=1}^{n} k|\Delta a_k| \int_0^{\frac{1}{n}} |D_k(x)|\, dx$$

$$+ \left(\frac{1}{n+1}\right) \sum_{k=1}^{n} |a_k| \int_0^{\frac{1}{n}} |D_k(x)|\, dx$$

or $$\int_0^{\frac{1}{n}} |f_n(x) - \sigma_n(x)|\, dx = O\left[\left(\frac{1}{n}\right)\sum_{k=1}^{n} k|\Delta a_k|\right], \quad n\to\infty .$$

Hence, for the first integral in (3.2) we have

$$\int_0^{\frac{1}{n}} |f_n(x) - f(x)|\, dx = O\left(\|\sigma_n(x) - f(x)\|\right) + O\left[\left(\frac{1}{n}\right)\sum_{k=1}^{n} k|\Delta a_k|\right], \quad n\to\infty .$$

For the second integral of right hand side of (3.2), we have

(3.3) $$\int_{\frac{1}{n}}^{\pi} |f_n(x) - f(x)|\, dx \le \int_{\frac{1}{n}}^{\pi} |a_{n+1}F_n(x)|\, dx + \int_{\frac{1}{n}}^{\pi} \left|\sum_{k=n+1}^{\infty} \Delta a_k D_k(x)\right| dx$$

$$\le \int_0^{\pi} |a_{n+1} F_n(x)|dx + \int_{\frac{1}{n}}^{\pi} \left|\sum_{k=n+1}^{\infty} \Delta a_k D_k(x)\right| dx$$

$$= (\pi/2)a_{n+1} + \int_{\frac{1}{n}}^{\pi} \left|\sum_{k=n+1}^{\infty} \Delta a_k D_k(x)\right| dx .$$

Combining all these estimates as in [2], we obtain

$$\int_0^{\pi} |f_n(x) - f(x)|dx = O\left(\|\sigma_n(x) - f(x)\| + \sum_{k=1}^{n} k|\Delta a_k| + \sum_{k=n+1}^{\infty} k^{p-1}|\Delta a_k|^p\right)$$

$$= o(1), \quad n\to\infty .$$

This completes the proof of our result.

Corollary. If $\{a_k\}$ belongs to F_p, then $\|S_n(x) - f(x)\| = o(1)$ if and only if $a_n \log n = o(1)$, $n\to\infty$.

Proof . We have

$$\int_0^{\pi} | f(x) - S_n(x) |\, dx = \int_0^{\pi} | f(x) - f_n(x) + f_n(x) - S_n(x) |\, dx$$

$$\le \int_0^{\pi} | f(x) - f_n(x) |dx + \int_0^{\pi} | f_n(x) - S_n(x) |dx$$

$$\le \int_0^{\pi} | f(x) - f_n(x) |dx + \int_0^{\pi} |a_{n+1} D_n(x)|dx + \int_{-\pi}^{\pi} |a_{n+1}K_n(x)|\, dx$$

and

$$\int_0^{\pi} |a_{n+1} D_n(x)|dx \leq \int_0^{\pi} |f_n(x) - S_n(x)|dx + \int_{-\pi}^{\pi} |a_{n+1}K_n(x)|dx$$

$$\leq \int_0^{\pi} |f(x) - S_n(x)|dx + \int_{-\pi}^{\pi} |a_{n+1}K_n(x)|dx .$$

Since $\int_{-\pi}^{\pi} |D_n(x)|dx$ behave like log n for large values of nand $\int_0^{\pi} |f(x) - f_n(x)|dx=0$ by our theorem for $n \to \infty$, the corollary follows.

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