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Flexure of Thick Cantilever Beam using Third Order Shear Deformation Theory

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Abstract:- The number of variables in the present theory is same as that in the first order shear deformation theory. A third order shear deformation theory for flexure of thick beams, taking into account transverse shear deformation effects, is developed. The sinusoidal function is used in displacement field in terms of thickness coordinate to represent the shear deformation effects. The noteworthy feature of this theory is that the transverse shear stresses can be obtained directly from the use of constitutive relations with excellent accuracy, satisfying the shear stress free conditions on the top and bottom surfaces of the beam. Hence, the theory obviates the need of shear correction factor. The thick cantilever isotropic beams are considered for the numerical studies to demonstrate the efficiency of theory. Results obtained are discussed critically with those of other theories.

Keywords:- Thick beam, displacements, stresses, , third order shear deformation, flexure.

I. INTRODUCTION

It is well-known that elementary theory of bending of beam based on Euler-Bernoulli hypothesis disregards the effects of the shear deformation and stress concentration. The theory is suitable for slender beams and is not suitable for thick or deep beams since it is based on the assumption that the transverse normal to neutral axis remains so during bending and after bending, implying that the transverse shear strain is zero. Since theory neglects the transverse shear deformation, it underestimates deflections in case of thick beams where shear deformation effects are significant.

Bresse [1], Rayleigh [2] and Timoshenko [3] were the pioneer investigators to include refined effects such as rotatory inertia and shear deformation in the beam theory. Timoshenko showed that the effect of transverse vibration of prismatic bars. This theory is now widely referred to as Timoshenko beam theory or first order shear deformation theory (FSDT) in the literature. In this theory transverse shear strain distribution is assumed to be constant through the beam thickness and thus requires shear correction factor to appropriately represent the strain energy of deformation. Cowper [4] has given refined expression for the shear correction factor for different cross-sections of beam. The accuracy of Timoshenko beam theory for transverse vibrations of cantilever beam in respect of the fundamental frequency is verified by Cowper [5] with a plane stress exact elasticity solution. To remove the discrepancies in classical and first order shear deformation theories, higher order or refined shear deformation theories were developed and are available in the open literature for static and vibration analysis of beam.

Levinson [6], Bickford [7], Rehfield and Murty [8], Krishna Murty [9], Baluch, Azad and Khidir [10], Bhimaraddi and Chandrashekhara [11] presented cosine shear deformation theories assuming a higher variation of axial displacement in terms of thickness coordinate. These theories satisfy shear stress free boundary conditions on top and bottom surfaces of beam and thus obviate the need of shear correction factor. Irretier [12] studied the refined dynamical effects in linear, homogenous beam according to theories, which exceed the limits of the Euler-Bernoulli beam theory. These effects are rotary inertia, shear deformation, rotary inertia and shear deformation, axial pre-stress, twist and coupling between bending and torsion.

Vlasov and Leont'ev [13] developed refined shear deformation theories for thick beams including sinusoidal function in terms of thickness coordinate in displacement field. However, with these theories shear stress free boundary conditions are not satisfied at top and bottom surfaces of the beam. In this paper development of theory and its application to thick cantilever beam is presented.

II. SYSTEM DEVELOPMENT

The beam under consideration as shown in Fig. 1 occupies in $0-x-y-z$ Cartesian coordinate system the region: $0 \le x \le L$; $0 \le y \le b$; $-\frac{h}{2} \le z \le \frac{h}{2}$

Where *x*, *y*, *z* are Cartesian coordinates, *L* and *b* are the length and width of beam in the x and *y* directions respectively, and *h* is the thickness of the beam in the z-direction. The beam is made up of homogeneous, linearly elastic isotropic material.

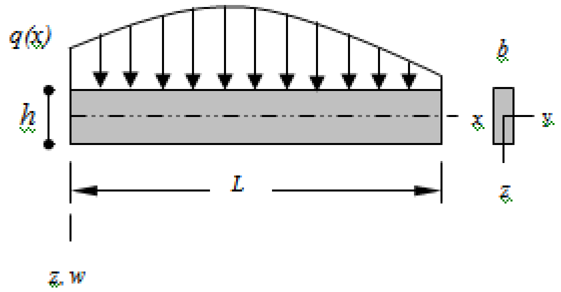

Fig.1: Beam under bending in *x-z* plane

A. *The Displacement Field*

The displacement field of the present beam theory is of the form:

$$u(x,z) = -z\frac{dw}{dx} + z\left(1-\frac{4}{3}\frac{z^2}{h^2}\right)\phi(x)$$
$$w(x,z) = w(x) \tag{1}$$

Where u is the axial displacement in x direction and w is the transverse displacement in z direction of the beam. The sinusoidal function is assigned according to the shear stress distribution through the thickness of the beam. The function ϕ represents rotation of the beam at neutral axis, which is an unknown function to be determined. The stress-strain relationships used are as follows:

$$\sigma_x = E\varepsilon_x,\ \tau_{zx} = G\gamma_{zx} \tag{2}$$

B. *Governing Equations and Boundary Conditions*

Using the principle of virtual work, variationally consistent governing differential equations and boundary conditions for the beam under consideration can be obtained. The principle of virtual work when applied to the beam leads to:

$$b\int_{x=0}^{x=L}\int_{z=-h/2}^{z=+h/2}\left(\sigma_x\delta\varepsilon_x + \tau_{zx}\delta\gamma_{zx}\right)dx\,dz - \int_{x=0}^{x=L} q(x)\delta w\,dx = 0 \tag{3}$$

where the symbol δ denotes the variational operator.

The associated consistent natural boundary conditions obtained are of following form:

At the ends *x* = 0 and *x* = *L*

$$V_x = EI\frac{d^3w}{dx^3} - \frac{24}{\pi^3}EI\frac{d^2\phi}{dx^2} = 0 \quad \text{or } w \text{ is prescribed} \tag{4}$$

$$M_x = EI\frac{d^2w}{dx^2} - \frac{24}{\pi^3}EI\frac{d\phi}{dx} = 0 \quad \text{or } \frac{dw}{dx} \text{ is prescribed} \tag{5}$$

$$M_a = EI\frac{24}{\pi^3}\frac{d^2w}{dx^2} - \frac{6}{\pi^2}EI\frac{d\phi}{dx} = 0 \quad \text{or } \phi \text{ is prescribed} \tag{6}$$

Thus the boundary value problem of the beam bending is given by the above variationally consistent governing differential equations and boundary conditions.

C. *The General Solution of Governing Equilibrium Equations of Beam*

The general solution for transverse displacement w(x) and warping function ϕ (x) is obtained using "(6) and (7)" using method of solution of linear differential equations with constant coefficients. Integrating and rearranging "(6)", we obtain the following expression

$$\frac{d^3w}{dx^3} = \frac{24}{\pi^3}\frac{d^2\phi}{dx^2} + \frac{Q(x)}{EI} \tag{7}$$

Where Q(x) is the generalized shear force for beam and it is given by $Q(x) = \int_0^x q\,dx + C_1$.

III. ILLUSTRATIVE EXAMPLE

In order to prove the efficacy of the present theory, the following numerical examples are considered. The following material properties for beam are used

E = 210 GPa, μ = 0.3 and ρ = 7800 kg/m^3, where E is the Young's modulus, ρ is the density, and μ is the Poisson's Units

Cantilever beam subjected to cosine load

The cantilever beam is having its origin at left at x = 0 and free at x =L. The beam is subjected to cosine load, on surface z = +h/2 acting in the downward z direction with maximum intensity of load q_0.

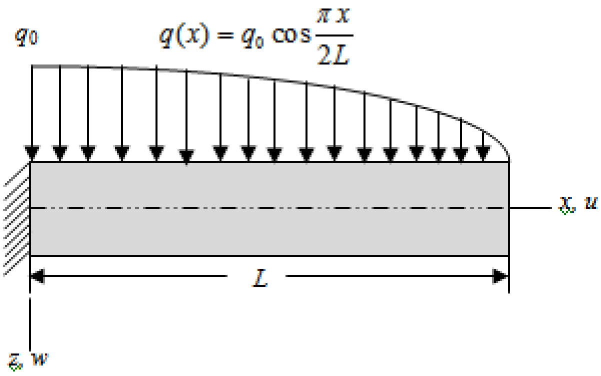

Fig.2: Cantilever beam with cosine load

General expressions obtained for $w(x)$ and $\phi(x)$ are as follows:

$$\phi(x)=\frac{2}{\pi}\frac{A_0}{C_0}\frac{q_0L}{Gbh}\left(1-\sin\frac{\pi x}{2L}+\sinh\lambda x-\cosh\lambda x\right) \tag{8}$$

$$w(x)=\frac{q_0L^4}{120EI}\left[\begin{aligned}&\frac{240}{\pi}\left(\frac{8}{\pi^3}\left(\cos\frac{\pi x}{2L}-1\right)-\frac{1}{6}\frac{x^3}{L^3}+\frac{1}{2}\frac{x^2}{L^2}\right)+\frac{40}{\pi^2}\frac{B_0}{C_0}\frac{E}{G}\frac{h^2}{L^2}\left(\cos\frac{\pi x}{2L}-1\right)\\&-\frac{20}{\pi}\frac{A_0^2}{C_0}\frac{E}{G}\frac{h^2}{L^2}\left(\frac{\sinh\lambda x-\cosh\lambda x+1}{\lambda L}-\frac{x}{L}\right)\end{aligned}\right] \tag{9}$$

The axial displacement and stresses obtained based on above solutions are as follows

$$u=\frac{q_0h}{Eb}\left\{\begin{aligned}&-\frac{1}{10}\frac{z}{h}\frac{L^3}{h^3}\left[\begin{aligned}&\frac{240}{\pi}\left(\frac{4}{\pi^2}\sin\frac{\pi x}{2L}\right)-\frac{1}{2}\frac{x^2}{L^2}+\frac{x}{L}-\frac{40}{\pi^2}\frac{B_0}{C_0}\frac{E}{G}\frac{h^2}{L^2}\left(\frac{\pi}{2}\sin\frac{\pi x}{2L}\right)\\&-\frac{20}{\pi}\frac{A_0^2}{C_0}\frac{E}{G}\frac{h^2}{L^2}\left(\cosh\lambda x-\sinh\lambda x-1\right)\end{aligned}\right]\\&+\frac{z}{h}\left[1-\frac{4}{3}\frac{z^2}{h^2}\right]\frac{2}{\pi}\frac{E}{G}\frac{A_0}{C_0}\frac{L}{h}\left(\sinh\lambda x-\cosh\lambda x+1-\sin\frac{\pi x}{2L}\right)\end{aligned}\right\} \tag{10}$$

$$\sigma_x=\frac{q_0}{b}\left\{\begin{aligned}&-\frac{1}{10}\frac{z}{h}\frac{L^2}{h^2}\left[\frac{240}{\pi}\left(-\frac{2}{\pi}\cos\frac{\pi x}{2L}+1-\frac{x}{L}\right)-10\frac{B_0}{C_0}\frac{E}{G}\frac{h^2}{L^2}\cos\frac{\pi x}{2L}-\frac{20}{\pi}\frac{A_0^2}{C_0}\frac{E}{G}\frac{h^2}{L^2}\left(\lambda L\sinh\lambda x-\lambda L\cosh\lambda x\right)\right]\\&+\frac{z}{h}\left[1-\frac{4}{3}\frac{z^2}{h^2}\right]\frac{2}{\pi}\frac{E}{G}\frac{A_0}{C_0}\left(-\frac{\pi}{2}\cos\frac{\pi x}{2L}+\lambda L\cosh\lambda x-\lambda L\sinh\lambda x\right)\end{aligned}\right\} \tag{11}$$

$$\tau_{zx}^{CR}=\frac{q_0}{b}\frac{2}{\pi}\frac{A_0}{C_0}\frac{L}{h}\left(1-4\frac{z^2}{h^2}\right)\left(1-\sin\frac{\pi x}{2L}+\sinh\lambda x-\cosh\lambda x\right) \tag{12}$$

$$\begin{aligned}\tau_{zx}^{EE}=&\frac{q_0L}{80bh}\left(1-4\frac{z^2}{h^2}\right)\left[\frac{240}{\pi}\left(1-\sin\frac{\pi x}{2L}\right)-5\pi\frac{B_0}{C_0}\frac{E}{G}\frac{h^2}{L^2}\sin\frac{\pi x}{2L}+\frac{20}{\pi}\frac{A_0^2}{C_0}\frac{E}{G}\frac{h^2}{L^2}\lambda^2L^2\left(\cosh\lambda x-\sinh\lambda x\right)\right].\\&-\frac{2}{\pi}\frac{E}{G}\frac{A_0}{C_0}\frac{h}{L}\left(\frac{1}{2}\frac{z^2}{h^2}-\frac{1}{3}\frac{z^4}{h^4}-\frac{5}{48}\right)\left(\lambda^2L^2\sinh\lambda x-\lambda^2L^2\cosh\lambda x+\frac{\pi^2}{4}\sin\frac{\pi x}{2L}\right)\end{aligned} \tag{13}$$

IV. RESULTS

In this paper, the results for inplane displacement, transverse displacement, inplane and transverse stresses are presented in the following non dimensional form for the purpose of presenting the results in this work.

For beams subjected to cosine load, *q(x)*

$$\bar{u}=\frac{Ebu}{q_0 h},\quad \bar{w}=\frac{10Ebh^3 w}{q_0 L^4},\bar{\sigma}_x=\frac{b\sigma_x}{q_0},\quad \bar{\tau}_{zx}=\frac{b\tau_{zx}}{q_0}$$

Table I: Non-Dimensional Axial Displacement at (*x* = *L*, *z* = *h*/2), Transverse Deflection at (*x* = *L*, *z* =0.0), Axial Stress at (x = 0, z = h/2) Maximum Transverse Shear Stresses (x = 0.01L, z = 0.0) for Aspect Ratio 4

Model	$\bar{u}$	$\bar{w}$	$\bar{\sigma}_x$	$\bar{\tau}_{zx}^{CR}$	$\bar{\tau}_{zx}^{EE}$
HSDT	-71.15	6.17	37.00	1.90	-2.76
FSDT	23.15	6.54	22.21	0.31	3.76
ETB	23.15	5.75	22.21	-	3.76

Table II: Non-Dimensional Axial Displacement at (*x* = *L*, *z* = *h*/2), Transverse Deflection at (x = L, z =0.0), Axial Stress at (*x* = 0, *z* = *h*/2) Maximum Transverse Shear Stresses (x = 0.01*L*, *z* = 0.0) for Aspect Ratio 10

Model	$\bar{u}$	$\bar{w}$	$\bar{\sigma}_x$	$\bar{\tau}_{zx}^{CR}$	$\bar{\tau}_{zx}^{EE}$
HSDT	-1064.34	5.82	171.47	7.82	3.86
FSDT	361.78	5.88	138.80	4.80	9.40
ETB	361.78	5.75	138.80	-	9.40

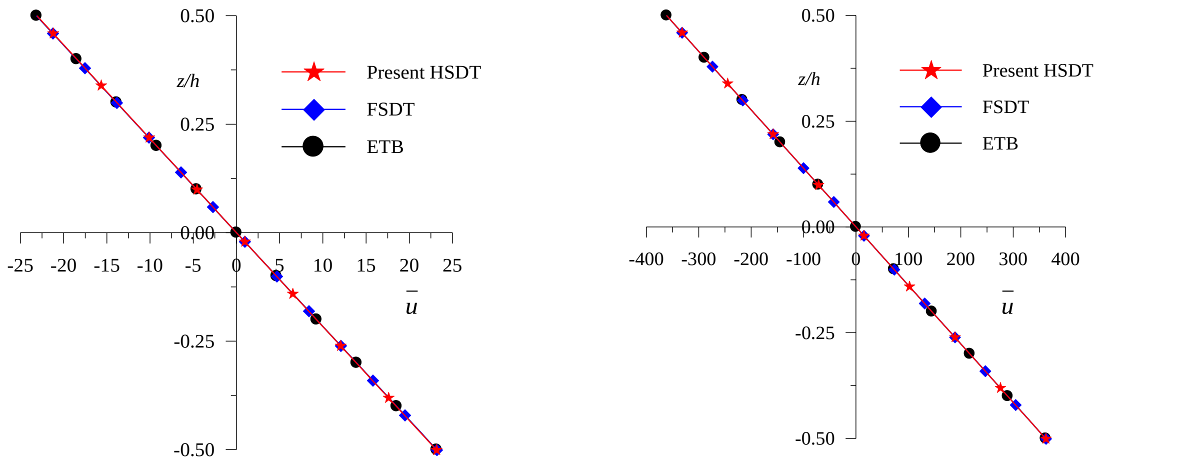

Fig. 3: Variation of axial displacement ($\bar{u}$) through the thickness of cantilever beam at (*x*=*L*, *z*) when subjected to cosine load for aspect ratio 4 and 10.

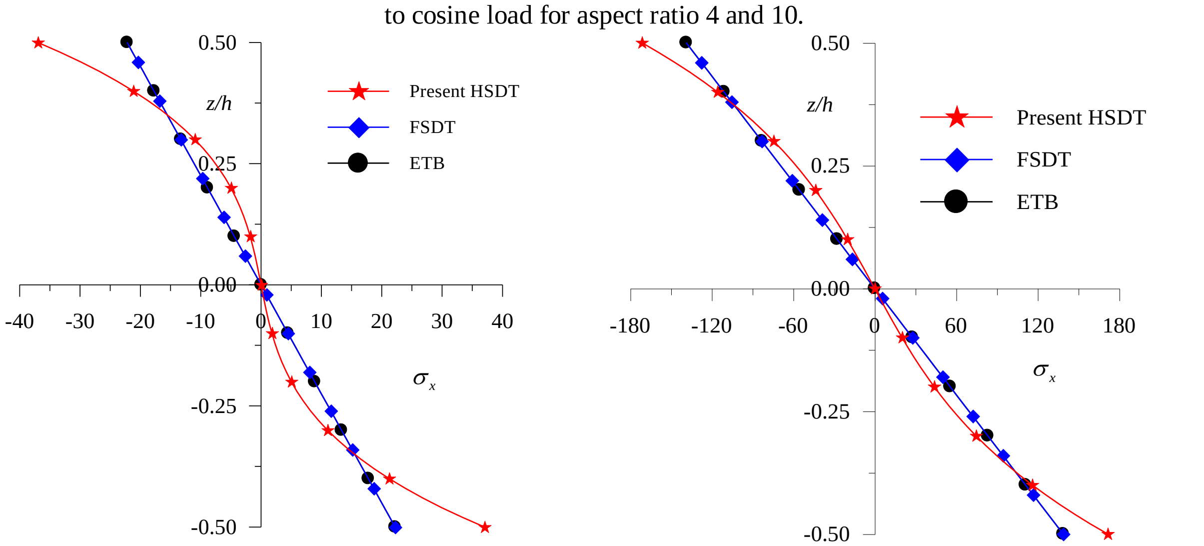

Fig. 4: Variation of axial stress (σ_x) through the thickness of cantilever beam at (*x* = 0, *z*) when subjected to cosine load for aspect ratio 4 and 10.

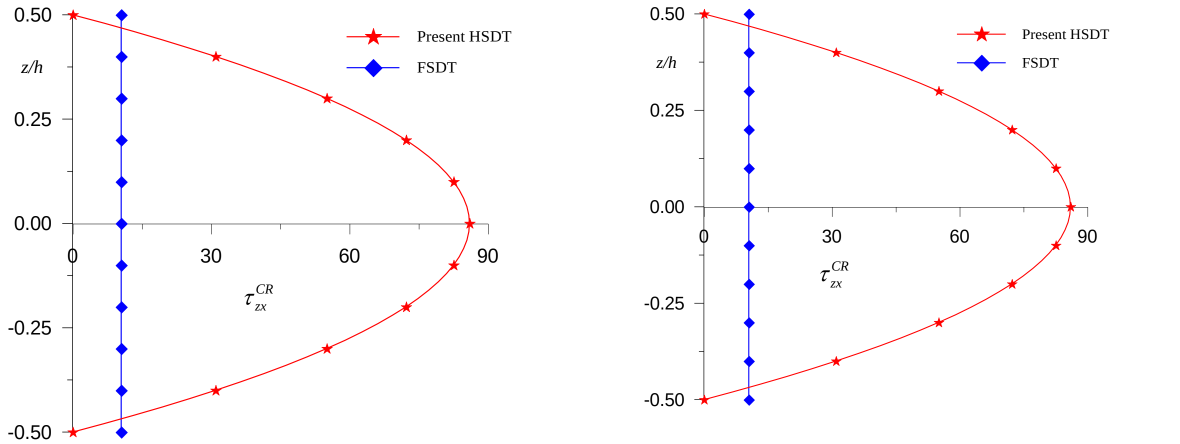

Fig. 5: Variation of transverse shear stress (τ_{zx}^{CR}) through the thickness of cantilever beam at (x = 0.01L, z) obtained via equilibrium equation when subjected to cosine load for aspect ratio 4 and 10.

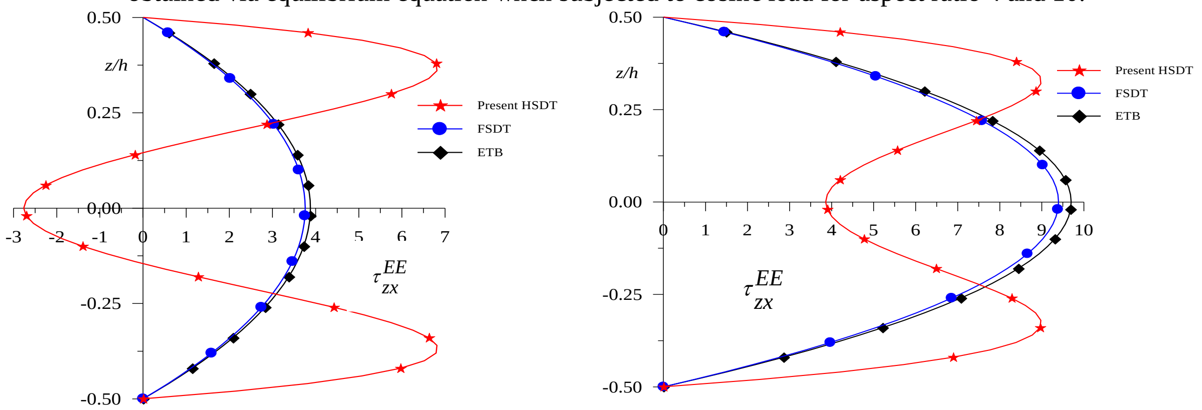

Fig. 6: Variation of transverse shear stress (τ_{zx}^{EE}) through the thickness of cantilever beam at (x = 0.01L, z) obtain via constitutive relation when subjected to cosine load for aspect ratio 4 and 10.

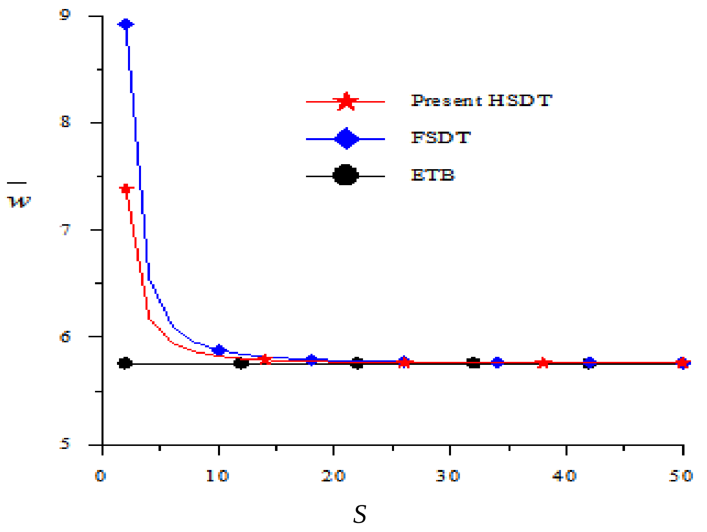

Fig. 7: Variation of displacement ($\bar{w}$) through the thickness of cantilever beam at (x = L, z) when subjected to cosine load for aspect ratio S.

V. DISCUSSION OF RESULTS

The comparison of results of maximum non-dimensional axial displacement ($\bar{u}$) for the aspect ratios of 4 and 10 is presented in Tables I and II for beam subjected to cosine loads. The values of axial displacement given by present theory are in close agreement with the values of other refined theories for aspect ratio 4 and 10.

The through thickness distribution of this displacement obtained by present theory is in close agreement with other refined theories as shown in Figs. 3 and 4 for aspect ratio 4 and 10.

The results of axial stress ($\bar{\sigma}_x$) are shown in Tables I and II for aspect ratios 4 and 10. The axial stresses given by present theory are compared with other higher order shear deformation theories. It is observed that the results by present theory are in excellent agreement with other refined theories as well as ETB and FSDT. The through the thickness variation of this stress given by all the theories is linear. The variations of this stress are shown in Figs. 5 and 6.

The comparison of maximum non-dimensional transverse shear stress for simple beams with varying and cosine loads obtained by the present theory and other refined theories is presented in Tables I and II for aspect ratio of 4 and 10 respectively. The maximum transverse shear stress obtained by present theory using constitutive relation is in good agreement with that of higher order theories for aspect ratio 4 and for aspect ratio 10. The through thickness variation of this stress obtained via constitutive relation are presented graphically in Figs. 7 and 8 and those obtained via equilibrium equation are presented in Figs. 9 and 10. The through thickness variation of this stress when obtained by various theories via equilibrium equation shows the excellent agreement with each other. The maximum value of this stress occurs at the neutral axis.

The comparison of results of maximum non-dimensional transverse displacement ($\bar{w}$) for the aspect ratios of 4 and 10 is presented in Tables I and II cantilever beam subjected to cosine load. The values of present theory are in excellent agreement with the values of other refined theories for aspect ratio 4 and 10 except those of classical beam theory (ETB) and FSDT of Timoshenko. The variation of $\bar{w}$ with aspect ratio (S) is shown in Fig. 11. The refined theories converge to the values of classical beam theory for the higher aspect ratios.

VI. CONCLUSION

The general solutions for beams with cosine loads are obtained in case of thick cantilever beams. The displacements and stresses obtained by present theory are in excellent agreement with those of other theories. The present theory yields the realistic variation of axial displacement and stresses through the thickness of beam. Thus the validity of the present theory is established.

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