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On the Zeros of Complex Polynomials in a Given Circle

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Abstract:- The problem of finding the number of zeros of a polynomial in a given circle is of great interest in the theory of the distribution of zeros of polynomials. In this paper we consider the said problem under certain conditions on the coefficients of the polynomial or their real and imaginary parts and prove certain results that generalize some well-known results on the subject.

Mathematics Subject Classification: 30C10, 30C15
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I. INTRODUCTION AND STATEMENT OF RESULTS

In the literature many results have been proved on the number of zeros of a polynomial in a given circle. Recently M. H. Gulzar [3] proved the following results:

Theorem A: : Let Let $P(z)=\sum_{j=0}^{\infty} a_j z^j$ be a polynomial o f degree n such that $\operatorname{Re}(a_j)=\alpha_j$, $\operatorname{Im}(a_j)=\beta_j$ and for some positive integers λ, μ and for some real numbers k_1, k_2, τ_1, τ_2,

$$k_1\alpha_n \le \alpha_{n-1} \le \ldots\ldots \le \alpha_{\lambda+1} \le \alpha_\lambda \ge \ldots\ldots \ge \alpha_1 \ge \tau_1\alpha_0$$

$$k_2\beta_n \le \beta_{n-1} \le \ldots\ldots \le \beta_{\mu+1} \le \beta_\mu \ge \ldots\ldots \ge \beta_1 \ge \tau_2\beta_0 .$$

Then the number of zeros of P(z) in $|z| \le \delta, 0<\delta<1$ does not exceed

$$\frac{1}{\log\frac{1}{\delta}}\log\frac{M^*}{|a_0|}, \text{ where}$$

$$M^* = |a_n| + |(k_1-1)\alpha_n| + |(k_2-1)\beta_n| + 2(\alpha_\lambda+\beta_\mu) - (k_1\alpha_n + k_2\beta_n) + |(\tau_1-1)\alpha_0|$$
$$-\tau_1\alpha_0 + |(\tau_2-1)\beta_0| - \tau_2\beta_0 + |a_0| .$$

Theorem B: Let Let $P(z)=\sum_{j=0}^{\infty} a_j z^j$ be a polynomial o f degree n such that $\operatorname{Re}(a_j)=\alpha_j$, $\operatorname{Im}(a_j)=\beta_j$ and for some positive integers $\lambda, \mu, \rho, \sigma$ and for some real numbers $k_1, k_2, k_3, k_4; \tau_1, \tau_2, \tau_3, \tau_4; 0<k_1\le 1, 0<k_2\le 1, 0<k_3\le 1, 0<k_4\le 1, 0<\tau_1\le 1, 0<\tau_2\le 1,$
$0<\tau_1\le 1, 0<\tau_2\le 1, 0<\tau_3\le 1, 0<\tau_4\le 1$,

$$k_1\alpha_{2[\frac{n}{2}]} \le \ldots\ldots \le \alpha_{2\lambda+2} \le \alpha_{2\lambda} \ge \ldots\ldots \ge \alpha_2 \ge \tau_1\alpha_0$$

$$k_2\alpha_{2[\frac{n}{2}]-1} \le \ldots\ldots \le \alpha_{2\mu+1} \le \alpha_{2\mu-1} \ge \ldots\ldots \ge \alpha_3 \ge \tau_2\alpha_1$$

$$k_3\beta_{2[\frac{n}{2}]} \le \ldots\ldots \le \beta_{2\rho+2} \le \beta_{2\rho} \ge \ldots\ldots \ge \beta_2 \ge \tau_3\beta_0$$

$$k_2\beta_{2[\frac{n}{2}]-1} \le \ldots\ldots \le \beta_{2\sigma+1} \le \beta_{2\sigma-1} \ge \ldots\ldots \ge \beta_3 \ge \tau_4\beta_1 .$$

.

If n is even, then P(z) has all its zeros in the disk $r_1 \le |z| \le r_2$, where

$$r_1 = \frac{|a_0|}{M} \quad \text{and} \quad r_2 = \frac{M'}{|a_n|},$$

with

$$M = |a_n| + |a_{n-1}| + |a_1| + |(k_1 - 1)\alpha_n| + |(k_2 - 1)\alpha_{n-1}| + |(k_3 - 1)\beta_n| + |(k_4 - 1)\beta_{n-1}|$$
$$+ 2(\alpha_{2\lambda} + \alpha_{2\mu-1} + \beta_{2\rho} + \beta_{2\sigma-1}) - (k_1\alpha_n + k_3\beta_n) - (k_2\alpha_{n-1} + k_4\beta_{n-1})$$
$$- (\tau_2\alpha_1 + \tau_4\beta_1) - (\tau_1\alpha_0 + \tau_3\beta_0) + |(\tau_1 - 1)\alpha_0| + |(\tau_2 - 1)\alpha_1| + |(\tau_3 - 1)\beta_0|$$
$$+ |(\tau_4 - 1)\beta_1|$$

and

$$M' = |a_{n-1}| + |a_1| + |a_0| + |(k_1 - 1)\alpha_n| + |(k_2 - 1)\alpha_{n-1}| + |(k_3 - 1)\beta_n| + |(k_4 - 1)\beta_{n-1}|$$
$$+ 2(\alpha_{2\lambda} + \alpha_{2\mu-1} + \beta_{2\rho} + \beta_{2\sigma-1}) - (k_1\alpha_n + k_3\beta_n) - (k_2\alpha_{n-1} + k_4\beta_{n-1})$$
$$- (\tau_2\alpha_1 + \tau_4\beta_1) - (\tau_1\alpha_0 + \tau_3\beta_0) + |(\tau_1 - 1)\alpha_0| + |(\tau_2 - 1)\alpha_1| + |(\tau_3 - 1)\beta_0|$$
$$+ |(\tau_4 - 1)\beta_1|.$$

If n is odd, then P(z) has all its zeros in the disk $R_1 \le |z| \le R_2$, where

$$R_1 = \frac{|a_0|}{M''} \quad \text{and} \quad R_2 = \frac{M'''}{|a_n|},$$

and M'' and M''' are respectively the same as M and M', except that k_1, k_2, k_3, k_4 are respectively replaced by k_2, k_1, k_4, k_3.

In this paper we shall find a bound for the number of zeros of the polynomials of Theorems A and B in a circle of any positive radius. In fact, we prove the following results:

Theorem 1: Let Let $P(z) = \sum_{j=0}^{\infty} a_j z^j$ be a polynomial o f degree n such that $\text{Re}(a_j) = \alpha_j$, $\text{Im}(a_j) = \beta_j$ and for some positive integers λ, μ and for some real numbers $k_1, k_2, \tau_1, \tau_2; 0 < k_1 \le 1, 0 < k_2 \le 1, 0 < \tau_1 \le 1, 0 < \tau_2 \le 1,$

$$k_1\alpha_n \le \alpha_{n-1} \le \ldots\ldots \le \alpha_{\lambda+1} \le \alpha_\lambda \ge \alpha_{\lambda-1} \ge \ldots\ldots \ge \alpha_1 \ge \tau_1\alpha_0$$
$$k_2\beta_n \le \beta_{n-1} \le \ldots\ldots \le \beta_{\mu+1} \le \beta_\mu \ge \beta_{\mu-1} \ge \ldots\ldots \ge \beta_1 \ge \tau_1\beta_0.$$

Then the number of zeros of P(z) in $|z| \le \frac{R}{c} \; (R > 0, c > 1)$ does not exceed $\frac{1}{\log c} \log \frac{M_1}{|a_0|}$, where

$$M_1 = |a_n|R^{n+1} + |a_0| + R^n[|\alpha_n| + |\beta_n| - k_1(|\alpha_n| + \alpha_n) - k_2(|\beta_n| + \beta_n) + \alpha_\lambda + \beta_\mu]$$
$$+ R^\lambda[\alpha_\lambda - \tau_1(|\alpha_0| + \alpha_0) + |\alpha_0|] + R^\mu[\beta_\mu - \tau_2(|\beta_0| + \beta_0) + |\beta_0|] \quad \text{for } R \ge 1$$

and

$$M_1 = |a_n|R^{n+1} + |a_0| + R^n[|\alpha_n| + |\beta_n| - k_1(|\alpha_n| + \alpha_n) - k_2(|\beta_n| + \beta_n) + \alpha_\lambda + \beta_\mu]$$
$$+ R[\alpha_\lambda + \beta_\mu - \tau_1(|\alpha_0| + \alpha_0) - \tau_2(|\beta_0| + \beta_0) + |\alpha_0| + |\beta_0|] \quad \text{for } R \le 1.$$

Remark 1: Taking R=1 and $c = \frac{1}{\delta}, 0 < \delta < 1$, Theorem 1 reduces to Theorem A.

For different values of k_1, k_2, τ_1, τ_2, Theorem 1 gives many other interesting results.

Theorem 2: Let Let $P(z) = \sum_{j=0}^{\infty} a_j z^j$ be a polynomial o f degree n such that $\text{Re}(a_j) = \alpha_j$, $\text{Im}(a_j) = \beta_j$ and for some positive integers $\lambda, \mu, \rho, \sigma$ and for some real numbers $k_1, k_2, k_3, k_4; \tau_1, \tau_2, \tau_3, \tau_4; 0 < k_1 \le 1, 0 < k_2 \le 1, 0 < k_3 \le 1, 0 < k_4 \le 1, 0 < \tau_1 \le 1, 0 < \tau_2 \le 1,$

$0<\tau_3\le 1, 0<\tau_4\le 1,$

$$k_1\alpha_{2[\frac{n}{2}]}\le .....\le \alpha_{2\lambda+2}\le \alpha_{2\lambda}\ge .....\ge \alpha_2\ge \tau_1\alpha_0$$

$$k_2\alpha_{2[\frac{n}{2}]-1}\le .....\le \alpha_{2\mu+1}\le \alpha_{2\mu-1}\ge .....\ge \alpha_3\ge \tau_2\alpha_1$$

$$k_3\beta_{2[\frac{n}{2}]}\le .....\le \beta_{2\rho+2}\le \beta_{2\rho}\ge .....\ge \beta_2\ge \tau_3\beta_0$$

$$k_2\beta_{2[\frac{n}{2}]-1}\le .....\le \beta_{2\sigma+1}\le \beta_{2\sigma-1}\ge .....\ge \beta_3\ge \tau_4\beta_1.$$

.

Then, for even n, the number of zeros of P(z) in in $|z|\le \frac{R}{c}(R>0, c>1)$ does not exceed $\frac{1}{\log c}\log\frac{M_2}{|a_0|}$,

where

$$M_2=|a_n|R^{n+1}+|a_0|+R^n[|\alpha_n|+|\beta_n|-k_1(|\alpha_n|+\alpha_n)-k_2(|\beta_n|+\beta_n)+\alpha_\lambda+\beta_\mu]$$
$$+R^\lambda[\alpha_\lambda-\tau_1(|\alpha_0|+\alpha_0)+|\alpha_0|]+R^\mu[\beta_\mu-\tau_2(|\beta_0|+\beta_0)+|\beta_0|] \text{ for } R\ge 1$$

and

$$M_2=|a_n|R^{n+1}+|a_0|+R^n[|\alpha_n|+|\beta_n|-k_1(|\alpha_n|+\alpha_n)-k_2(|\beta_n|+\beta_n)+\alpha_\lambda+\beta_\mu]$$
$$+R[\alpha_\lambda+\beta_\mu-\tau_1(|\alpha_0|+\alpha_0)-\tau_2(|\beta_0|+\beta_0)+|\alpha_0|+|\beta_0|] \text{ for } R\le 1.$$

If n is odd, the number of zeros of P(z) in $|z|\le \frac{R}{c}(R>0, c>1)$ does not exceed $\frac{1}{\log c}\log\frac{M_2'}{|a_0|}$, where

M_2' is same as M_2 except that k_1,k_2,k_3,k_4 are respectively replaced by k_2,k_1,k_4,k_3.

For different values of k_1,k_2,τ_1,τ_2, Theorem 2 gives many interesting results.

Theorem 3: Let Let $P(z)=\sum_{j=0}^{\infty}a_jz^j$ be a polynomial o f degree n such that for some positive integers $\lambda,\mu,$ and for some real numbers $k_1,k_2,\tau_1,\tau_2,\ \ 0<k_1\le 1, 0<k_2\le 1, 0<\tau_1\le 1, 0<\tau_2\le 1,$

$$k_1\left|a_{2[\frac{n}{2}]}\right|\le .....\le |a_{2\lambda+2}|\le |a_{2\lambda}|\ge .....\ge |a_2|\ge \tau_1|a_0|$$

$$k_2\left|a_{2[\frac{n}{2}]-1}\right|\le .....\le |a_{2\mu+1}|\le |a_{2\mu-1}|\ge .....\ge |a_3|\ge \tau_2|a_1|.$$

Then for even n the number of zeros of P(z) in $|z|\le \frac{R}{c}(R>0, c>1)$ does not exceed $\frac{1}{\log c}\log\frac{M_3}{|a_0|}$,

where for $R\ge 1$,

$$M_3=|a_n|R^{n+2}+|a_{n-1}|R^{n+1}+|a_0|+R^n[|a_1|+(1-k_1)|a_n|+(1-k_2)|a_{n-1}|$$
$$+\cos\alpha(2|a_{2\lambda}|+2|a_{2\mu}|-k_1|a_n|-k_2|a_{n-1}|-\tau_2|a_1|-\tau_1|a_0|)$$
$$+\sin\alpha(k_1|a_n|+k_2|a_{n-1}|+\tau_2|a_1|+\tau_1|a_0|+2\sum_{j=2}^{n-2}|a_j|)]$$

and for $R\le 1$,

$$M_3=|a_n|R^{n+2}+|a_{n-1}|R^{n+1}+|a_0|+R[|a_1|+(1-k_1)|a_n|+(1-k_2)|a_{n-1}|$$

$$+\cos\alpha(2|a_{2\lambda}|+2|a_{2\mu}|-k_1|a_n|-k_2|a_{n-1}|-\tau_2|a_1|-\tau_1|a_0|)$$
$$+\sin\alpha(k_1|a_n|+k_2|a_{n-1}|+\tau_2|a_1|+\tau_1|a_0|+2\sum_{j=2}^{n-2}|a_j|)].$$

If n is odd, then the number of zeros of P(z) in $|z|\le\frac{R}{c}(R>0,c>1)$ does not exceed $\frac{1}{\log c}\log\frac{M_4}{|a_0|}$, where M_4 is same as M_3 except that k_1,k_2 are respectively replaced by k_2,k_1.

For different values of k_1,k_2, Theorem 3 gives many interesting results.

II. LEMMAS

For the proofs of the above results we need the following results:

Lemma 1: If f(z) is analytic in $|z|\le R$,but not identically zero, f(0)$\neq$ 0 and $f(a_k)=0, k=1,2,......, n$, then

$$\frac{1}{2\pi}\int_0^{2\pi}\log|f(\mathrm{Re}^{i\theta}|d\theta-\log|f(0)|=\sum_{j=1}^{n}\log\frac{R}{|a_j|}$$

Lemma 1 is the famous Jensen's theorem (see page 208 of [1]).

Lemma 2: If f(z) is analytic and $|f(z)|\le M(r)$ in $|z|\le r$, then the number of zeros of f(z) in $|z|\le\frac{r}{c}, c>1$ does not exceed

$$\frac{1}{\log c}\log\frac{M(r)}{|f(0)|}.$$

Lemma 2 is a simple deduction from Lemma 1.

Lemma 3: Let $P(z)=\sum_{j=0}^{\infty}a_jz^j$ be a polynomial o f degree n with complex coefficients such that for some real α,β, $|\arg a_j-\beta|\le\alpha\le\frac{\pi}{2}, 0\le j\le n,$ and

$|a_j|\ge|a_{j-1}|, 0\le j\le n,$ then for any t>0,

$$|ta_j-a_{j-1}|\le(t|a_j|-|a_{j-1}|)\cos\alpha+(t|a_j|+|a_{j-1}|)\sin\alpha.$$

Lemma 3 is due to Govil and Rahman [2].

III. PROOFS OF THEOREMS

Proof of Theorem 1: Consider the polynomial

$$\begin{aligned}F(z)&=(1-z)P(z)\\&=(1-z)(a_nz^n+a_{n-1}z^{n-1}+......+a_1z+a_0)\\&=-a_nz^{n+1}+(a_n-a_{n-1})z^n+......+(a_1-a_0)z+a_0\\&=-a_nz^{n+1}+a_0-(k_1-1)\alpha_nz^n+(k_1\alpha_n-\alpha_{n-1})z^n+(\alpha_{n-1}-\alpha_{n-2})z^{n-1}\\&\quad+......+(\alpha_{\lambda+1}-\alpha_\lambda)z^{\lambda+1}+(\alpha_\lambda-\alpha_{\lambda-1})z^\lambda+.......+(\alpha_1-\tau_1\alpha_0)z\\&\quad+(\tau_1-1)\alpha_0z+i\{-(k_2-1)\beta_nz^n+(k_2\beta_n-\beta_{n-1})z^n+(\beta_{n-1}-\beta_{n-2})z^{n-1}+......\\&\quad+(\beta_{\mu+1}-\beta_\mu)z^{\mu+1}+(\beta_\mu-\beta_{\mu-1})z^\mu+......+(\beta_1-\tau_2\beta_0)z+(\tau_2-1)\beta_0z\}\end{aligned}$$

For $|z|\le R$, we have by using the hypothesis

$$|F(z)|\le|a_n|R^{n+1}+|a_0|+(1-k_1)|\alpha_n|R^n+(\alpha_{n-1}-k_1\alpha_n)R^n+(\alpha_{n-2}-\alpha_{n-1})R^{n-1}+......$$

$$+(\alpha_\lambda-\alpha_{\lambda+1})R^{\lambda+1}+(\alpha_\lambda-\alpha_{\lambda-1})R^{\lambda}+\ldots\ldots+(\alpha_1-\tau_1\alpha_0)R+(1-\tau_1)|\alpha_0|R$$
$$+(1-k_2)|\beta_n|R^n+(\beta_{n-1}-k_2\beta_n)R^n+(\beta_{n-2}-\beta_{n-1})R^{n-1}+\ldots\ldots$$
$$+(\beta_\mu-\beta_{\mu+1})R^{\mu+1}+(\beta_\mu-\beta_{\mu-1})R^{\mu}+\ldots\ldots+(\beta_1-\tau_2\beta_0)R+(1-\tau_2)|\beta_0|R$$

For $R\geq 1$,

$$|F(z)|\leq|a_n|R^{n+1}+|a_0|+R^n[(1-k_1)|\alpha_n|+\alpha_{n-1}-k_1\alpha_n+\alpha_{n-2}-\alpha_{n-1}+\ldots\ldots$$
$$+\alpha_{\lambda+2}-\alpha_{\lambda+1}+\alpha_\lambda-\alpha_{\lambda+1}+(1-k_2)|\beta_n|+\beta_{n-1}-k_2\beta_n+\beta_{n-2}-\beta_{n-1}+\ldots\ldots$$
$$+\beta_{\mu+2}-\beta_{\mu+1}+\beta_\mu-\beta_{\mu+1}]+R^{\lambda}[\alpha_\lambda-\alpha_{\lambda-1}+\alpha_{\lambda-1}-\alpha_{\lambda-2}+\ldots\ldots$$
$$+\alpha_1-\tau_1\alpha_{0\,2}+(1-\tau_1)|\alpha_0|]+R^{\mu}[\beta_\mu-\beta_{\mu-1}+\beta_{\mu-1}-\beta_{\mu-2}+\ldots\ldots$$
$$+\beta_1-\tau_2\beta_0+(1-\tau_2)|\beta_0|]$$
$$=|a_n|R^{n+1}+|a_0|+R^n[|\alpha_n|+|\beta_n|-k_1(|\alpha_n|+\alpha_n)-k_2(|\beta_n|+\beta_n)+\alpha_\lambda+\beta_\mu]$$
$$+R^{\lambda}[\alpha_\lambda-\tau_1(|\alpha_0|+\alpha_0)+|\alpha_0|]+R^{\mu}[\beta_\mu-\tau_2(|\beta_0|+\beta_0)+|\beta_0|]$$

For $R\leq 1$,

$$|F(z)|\leq|a_n|R^{n+1}+|a_0|+R^n[|\alpha_n|+|\beta_n|-k_1(|\alpha_n|+\alpha_n)-k_2(|\beta_n|+\beta_n)+\alpha_\lambda+\beta_\mu]$$
$$+R[\alpha_\lambda+\beta_\mu-\tau_1(|\alpha_0|+\alpha_0)-\tau_2(|\beta_0|+\beta_0)+|\alpha_0|+|\beta_0|].$$

Hence by Lemma 2, the number of zeros of F(z) and therefore P(z) in $|z|\leq\frac{R}{c}\,(R>0,c>1)$ does not exceed

$\frac{1}{\log c}\log\frac{M_1}{|a_0|}$, where

$$M_1=|a_n|R^{n+1}+|a_0|+R^n[|\alpha_n|+|\beta_n|-k_1(|\alpha_n|+\alpha_n)-k_2(|\beta_n|+\beta_n)+\alpha_\lambda+\beta_\mu]$$
$$+R^{\lambda}[\alpha_\lambda-\tau_1(|\alpha_0|+\alpha_0)+|\alpha_0|]+R^{\mu}[\beta_\mu-\tau_2(|\beta_0|+\beta_0)+|\beta_0|]\ \text{ for } R\geq 1$$

and

$$M_1=|a_n|R^{n+1}+|a_0|+R^n[|\alpha_n|+|\beta_n|-k_1(|\alpha_n|+\alpha_n)-k_2(|\beta_n|+\beta_n)+\alpha_\lambda+\beta_\mu]$$
$$+R[\alpha_\lambda+\beta_\mu-\tau_1(|\alpha_0|+\alpha_0)-\tau_2(|\beta_0|+\beta_0)+|\alpha_0|+|\beta_0|]\ \text{ for } R\leq 1.$$

That proves Theorem 1.

Proof of Theorem 2: Let n be even. Consider the polynomial

$$F(z)=(1-z^2)P(z)=(1-z^2)(a_nz^n+a_{n-1}z^{n-1}+\ldots\ldots+a_1z+a_0)$$
$$=-az^{n+2}-a_{n-1}z^{n+1}+(a_n-a_{n-2})z^n+(a_{n-1}-a_{n-3})z^{n-1}+\ldots\ldots+(a_3-a_1)z^3$$
$$+(a_2-a_0)z^2+a_1z+a_0$$
$$=-a_nz^{n+2}-a_{n-1}z^{n+1}-(k_1-1)\alpha_nz^n+(k_1\alpha_n-\alpha_{n-2})z^n-(k_2-1)z^{n-1}$$
$$+(k_2\alpha_{n-1}-\alpha_{n-3})z^{n-1}+(\alpha_{n-2}-\alpha_{n-4})z^{n-2}+(\alpha_{n-3}-\alpha_{n-5})z^{n-3}+\ldots\ldots$$
$$+(\alpha_3-\tau_2\alpha_1)z^3+(\tau_2\alpha_1-\alpha_1)z^3+(\alpha_2-\tau_1\alpha_0)z^2+(\tau_1\alpha_0-\alpha_0)z^2$$
$$+a_1z+a_0+i\{-(k_3-1)\beta_nz^n+(k_3\beta_n-\beta_{n-2})z^n-(k_4-1)\beta_{n-1}z^{n-1}$$
$$+(k_4\beta_{n-1}-\beta_{n-3})z^{n-1}+(\beta_{n-2}-\beta_{n-4})z^{n-2}+(\beta_{n-3}-\beta_{n-5})z^{n-3}+\ldots\ldots$$
$$+(\beta_3-\tau_4\beta_1)z^3+(\tau_4\beta_1-\beta_1)z^3+(\beta_2-\tau_3\beta_0)z^2+(\tau_3\beta_0-\beta_0)z^2.$$

For $|z|\leq R$, we have by using the hypothesis

$$|F(z)|\leq|a_n|R^{n+2}+|a_{n-1}|R^{n+1}+(1-k_1)|\alpha_n|R^n+(\alpha_{n-2}-k_1\alpha_n)R^n+(1-k_2)|\alpha_{n-1}|R^{n-1}$$

$$+(\alpha_{n-3}-k_2\alpha_{n-1})R^{n-1}+(\alpha_{n-4}-\alpha_{n-2})R^{n-2}+(\alpha_{n-5}-\alpha_{n-3})R^{n-3}+\ldots\ldots$$
$$+(\alpha_{2\lambda}-\alpha_{2\lambda+2})R^{2\lambda+2}+(\alpha_{2\lambda}-\alpha_{2\lambda-2})R^{2\lambda}\ldots\ldots+(\alpha_{2\mu-1}-\alpha_{2\mu+1})R^{2\mu+1}$$
$$+(\alpha_{2\mu-1}-\alpha_{2\mu-3})R^{2\mu-1}+\ldots\ldots+(\alpha_3-\tau_2\alpha_1)R^3+(1-\tau_2)|\alpha_1|R^3$$
$$+(\alpha_2-\tau_1\alpha_0)R^2+(1-\tau_1)|\alpha_0|R^2+|a_1|R+|a_0|+(1-k_3)|\beta_n|R^n$$
$$+(\beta_{n-2}-k_3\beta_n)R^n+\ldots\ldots+(\beta_{2\rho}-\beta_{2\rho+2})R^{2\rho+2}+(\beta_{2\rho}-\beta_{2\rho-2})R^{2\rho}$$
$$+\ldots\ldots+(\beta_{2\sigma-1}-\beta_{2\sigma+1})R^{2\sigma+1}+(\beta_{2\sigma-1}-\beta_{2\sigma-3})R^{2\sigma-1}+(\beta_3-\tau_4\beta_1)R^3$$
$$+(1-\tau_4)|\beta_1|R^3+(\beta_2-\tau_3\beta_0)R^2+(1-\tau_3)|\beta_0|R^2$$

For $R\geq 1$,

$$|F(z)|\leq|a_n|R^{n+2}+|a_{n-1}|R^{n+1}+|a_0|+R^n[(1-k_1)|\alpha_n|+\alpha_{n-2}-k_1\alpha_n+(1-k_2)|\alpha_{n-1}|$$
$$+\alpha_{n-3}-k_2\alpha_{n-1}+\alpha_{n-4}-\alpha_{n-2}+\alpha_{n-5}-\alpha_{n-3}+\ldots\ldots$$
$$+\alpha_{2\lambda}-\alpha_{2\lambda+2}+\alpha_{2\lambda}-\alpha_{2\lambda-2}\ldots\ldots+\alpha_{2\mu-1}-\alpha_{2\mu+1}$$
$$+\alpha_{2\mu-1}-\alpha_{2\mu-3}+\ldots\ldots+\alpha_3-\tau_2\alpha_1+(1-\tau_2)|\alpha_1|$$
$$+(\alpha_2-\tau_1\alpha_0)+(1-\tau_1)|\alpha_0|+|a_0|+(1-k_3)|\beta_n|$$
$$+\beta_{n-2}-k_3\beta_n+\ldots\ldots+\beta_{2\rho}-\beta_{2\rho+2}+\beta_{2\rho}-\beta_{2\rho-2}$$
$$+\ldots\ldots+\beta_{2\sigma-1}-\beta_{2\sigma+1}+\beta_{2\sigma-1}-\beta_{2\sigma-3}+\beta_3-\tau_4\beta_1$$
$$+(1-\tau_4)|\beta_1|+(\beta_2-\tau_3\beta_0)+(1-\tau_3)|\beta_0|+|a_1|]$$
$$=|a_n|R^{n+2}+|a_{n-1}|R^{n+1}+|a_0|+R^n[|\alpha_n|+|\beta_n|+|\alpha_{n-1}|+|\beta_{n-1}|$$
$$+2(\alpha_{2\lambda}+\alpha_{2\mu-1}+\beta_{2\rho}+\beta_{2\sigma-1})-(k_1|\alpha_n|+k_3|\beta_n|)$$
$$-(k_2|\alpha_{n-1}|+k_4|\beta_{n-1}|)-(\tau_2\alpha_1+\tau_4\beta_1)-(\tau_1\alpha_0+\tau_3\beta_0)$$
$$+(1-\tau_2)|\alpha_1|+(1-\tau_1)|\alpha_0|+(1-\tau_3)|\beta_1|(1-\tau_4)|\beta_0|+|a_1|].$$

For $R\leq 1$,

$$|F(z)|\leq|a_n|R^{n+2}+|a_{n-1}|R^{n+1}+|a_0|+R[|\alpha_n|+|\beta_n|+|\alpha_{n-1}|+|\beta_{n-1}|$$
$$+2(\alpha_{2\lambda}+\alpha_{2\mu-1}+\beta_{2\rho}+\beta_{2\sigma-1})-(k_1|\alpha_n|+k_3|\beta_n|)$$
$$-(k_2|\alpha_{n-1}|+k_4|\beta_{n-1}|)-(\tau_2\alpha_1+\tau_4\beta_1)-(\tau_1\alpha_0+\tau_3\beta_0)$$
$$+(1-\tau_2)|\alpha_1|+(1-\tau_1)|\alpha_0|+(1-\tau_3)|\beta_1|(1-\tau_4)|\beta_0|+|a_1|].$$

Hence, by Lemma 2, the number of zeros of F(z) and therefore P(z) in $|z|\leq\dfrac{R}{c}\,(R>0,c>1)$ does not exceed $\dfrac{1}{\log c}\log\dfrac{M_2}{|a_0|}$, where for $R\geq 1$,

$$M_2=|a_n|R^{n+2}+|a_{n-1}|R^{n+1}+|a_0|+R^n[|\alpha_n|+|\beta_n|+|\alpha_{n-1}|+|\beta_{n-1}|$$
$$+2(\alpha_{2\lambda}+\alpha_{2\mu-1}+\beta_{2\rho}+\beta_{2\sigma-1})-(k_1|\alpha_n|+k_3|\beta_n|)$$
$$-(k_2|\alpha_{n-1}|+k_4|\beta_{n-1}|)-(\tau_2\alpha_1+\tau_4\beta_1)-(\tau_1\alpha_0+\tau_3\beta_0)$$
$$+(1-\tau_2)|\alpha_1|+(1-\tau_1)|\alpha_0|+(1-\tau_3)|\beta_1|(1-\tau_4)|\beta_0|+|a_1|],$$

and for $R\leq 1$,

$$M_2=|a_n|R^{n+2}+|a_{n-1}|R^{n+1}+|a_0|+R[|\alpha_n|+|\beta_n|+|\alpha_{n-1}|+|\beta_{n-1}|$$

$$+2(\alpha_{2\lambda}+\alpha_{2\mu-1}+\beta_{2\rho}+\beta_{2\sigma-1})-(k_1|\alpha_n|+k_3|\beta_n|)$$
$$-(k_2|\alpha_{n-1}|+k_4|\beta_{n-1}|)-(\tau_2\alpha_1+\tau_4\beta_1)-(\tau_1\alpha_0+\tau_3\beta_0)$$
$$+(1-\tau_2)|\alpha_1|+(1-\tau_1)|\alpha_0|+(1-\tau_3)|\beta_1|(1-\tau_4)|\beta_0|+|a_1|].$$

The proof for odd n is similar and is omitted.

That proves Theorem 2.

Proof of Theorem 3: Suppose that n is even and the coefficient conditions hold i.e.

$$k_1|a_n|\le......\le|a_{2\lambda+2}|\le|a_{2\lambda}|\ge......\ge|a_2|\ge\tau_1|a_0|$$
$$k_2|a_{n-1}|\le......\le|a_{2\mu+1}|\le|a_{2\mu-1}|\ge......\ge|a_3|\ge\tau_2|a_1|.$$

Consider the polynomial

$$F(z)=(1-z^2)P(z)=(1-z^2)(a_nz^n+a_{n-1}z^{n-1}+......+a_1z+a_0)$$
$$=-az^{n+2}-a_{n-1}z^{n+1}+(a_n-a_{n-2})z^n+(a_{n-1}-a_{n-3})z^{n-1}+......+(a_3-a_1)z^3$$
$$+(a_2-a_0)z^2+a_1z+a_0$$
$$=-a_nz^{n+2}-a_{n-1}z^{n+1}-(k_1-1)a_nz^n+(k_1a_n-a_{n-2})z^n-(k_2-1)a_nz^{n-1}$$
$$+(k_2a_{n-1}-a_{n-3})z^{n-1}+(a_{n-2}-a_{n-4})z^{n-2}+(a_{n-3}-a_{n-5})z^{n-3}+\ldots\ldots$$
$$+(a_{2\lambda+2}-a_{2\lambda})z^{2\lambda+2}+(a_{2\lambda}-a_{2\lambda-2})z^{2\lambda}+\ldots\ldots+(a_{2\mu+1}-a_{2\mu-1})z^{2\mu+1}$$
$$+(a_{2\mu-1}-a_{2\mu-3})z^{2\mu-1}+\ldots\ldots+(a_3-\tau_2a_1)z^3+(\tau_2a_1-a_1)z^3$$
$$+(a_2-\tau_1a_0)z^2+(\tau_1a_0-a_0)z^2+a_1z+a_0$$

For $|z|\le R$, we have by using the hypothesis and Lemma 3

$$|F(z)|\le|a_n|R^{n+2}+|a_{n-1}|R^{n+1}+(1-k_1)|a_n|R^n+|a_{n-2}-k_1a_n|R^n+(1-k_2)|a_{n-1}|R^{n-1}$$
$$+|a_{n-3}-k_2a_{n-1}|R^{n-1}+|a_{n-4}-a_{n-2}|R^{n-2}+|a_{n-5}-a_{n-3}|R^{n-3}+\ldots\ldots$$
$$+|a_{2\lambda}-a_{2\lambda+2}|R^{2\lambda+2}+|a_{2\lambda}-a_{2\lambda-2}|R^{2\lambda}\ldots\ldots+|a_{2\mu-1}-a_{2\mu+1}|R^{2\mu+1}$$
$$+|a_{2\mu-1}-a_{2\mu-3}|R^{2\mu-1}+\ldots\ldots+|a_3-\tau_2a_1|R^3+(1-\tau_2)|a_1|R^3$$
$$+|a_2-\tau_1a_0|R^2+(1-\tau_1)|\alpha_0|R^2+|a_1|R+|a_0|$$
$$\le|a_n|R^{n+2}+|a_{n-1}|R^{n+1}+|a_0|+R^n[(1-k_1)|a_n|+(1-k_2)|a_{n-1}|$$
$$+(|a_{n-2}|-k_1|a_n|)\cos\alpha+(|a_{n-2}|+k_1|a_n|)\sin\alpha+(|a_{n-4}|-|a_{n-2}|)\cos\alpha$$
$$+(|a_{n-4}|+|a_{n-2}|)\sin\alpha+\ldots\ldots+(|a_{2\lambda}|-|a_{2\lambda+2}|)\cos\alpha+(|a_{2\lambda}|+|a_{2\lambda+2}|)\sin\alpha$$
$$+(|a_{2\lambda}|-|a_{2\lambda-2}|)\cos\alpha+(|a_{2\lambda}|+|a_{2\lambda-2}|)\sin\alpha+\ldots\ldots+(|a_{2\mu-1}|-|a_{2\mu+1}|)\cos\alpha$$
$$+(|a_{2\mu-1}|+|a_{2\mu+1}|)\sin\alpha+(|a_{2\mu-1}|-|a_{2\mu-3}|)\cos\alpha+(|a_{2\mu-1}|+|a_{2\mu-3}|)\sin\alpha$$
$$+\ldots\ldots+(|a_3|-\tau_2|a_1|)\cos\alpha+(|a_3|+\tau_2|a_1|)\sin\alpha+(1-\tau_2)|a_1|+(1-\tau_1)|a_0|$$
$$+(|a_2|-\tau_1|a_0|)\cos\alpha+(|a_2|+\tau_1|a_0|)\sin\alpha+|a_1|]$$
$$=|a_n|R^{n+2}+|a_{n-1}|R^{n+1}+|a_0|+R^n[|a_1|+(1-k_1)|a_n|+(1-k_2)|a_{n-1}|$$
$$+\cos\alpha(2|a_{2\lambda}|+2|a_{2\mu}|-k_1|a_n|-k_2|a_{n-1}|-\tau_2|a_1|-\tau_1|a_0|)$$
$$+\sin\alpha(k_1|a_n|+k_2|a_{n-1}|+\tau_2|a_1|+\tau_1|a_0|+2\sum_{j=2}^{n-2}|a_j|)]$$

for $R\ge 1$.

For $R\le 1$,

$$|F(z)| \le |a_n|R^{n+2} + |a_{n-1}|R^{n+1} + |a_0| + R[|a_1| + (1-k_1)|a_n| + (1-k_2)|a_{n-1}|$$
$$+ \cos\alpha(2|a_{2\lambda}| + 2|a_{2\mu}| - k_1|a_n| - k_2|a_{n-1}| - \tau_2|a_1| - \tau_1|a_0|)$$
$$+ \sin\alpha(k_1|a_n| + k_2|a_{n-1}| + \tau_2|a_1| + \tau_1|a_0| + 2\sum_{j=2}^{n-2}|a_j|)].$$

Hence , by Lemma 2, the number of zeros of F(z) and therefore P(z) in $|z| \le \frac{R}{c}(R>0, c>1)$ does not exceed $\frac{1}{\log c}\log\frac{M_3}{|a_0|}$, where for $R \ge 1$,

$$M_3 = |a_n|R^{n+2} + |a_{n-1}|R^{n+1} + |a_0| + R^n[|a_1| + (1-k_1)|a_n| + (1-k_2)|a_{n-1}|$$
$$+ \cos\alpha(2|a_{2\lambda}| + 2|a_{2\mu}| - k_1|a_n| - k_2|a_{n-1}| - \tau_2|a_1| - \tau_1|a_0|)$$
$$+ \sin\alpha(k_1|a_n| + k_2|a_{n-1}| + \tau_2|a_1| + \tau_1|a_0| + 2\sum_{j=2}^{n-2}|a_j|)]$$

and for $R \le 1$,

$$M_3 = |a_n|R^{n+2} + |a_{n-1}|R^{n+1} + |a_0| + R[|a_1| + (1-k_1)|a_n| + (1-k_2)|a_{n-1}|$$
$$+ \cos\alpha(2|a_{2\lambda}| + 2|a_{2\mu}| - k_1|a_n| - k_2|a_{n-1}| - \tau_2|a_1| - \tau_1|a_0|)$$
$$+ \sin\alpha(k_1|a_n| + k_2|a_{n-1}| + \tau_2|a_1| + \tau_1|a_0| + 2\sum_{j=2}^{n-2}|a_j|)].$$

That proves the result for even n. The case when n is odd is similar and is omitted.

REFERENCES

[1]. L.V.Ahlfors, Complex Analysis, 3rd edition, Mc-Grawhill

[2]. N. K. Govil and Q. I. Rahman, On the Enestrom- Kakeya Theorem, Tohoku Math. J. 20(1968),126-136.

[3]. M. H. Gulzar, Some Results on the Zeros of Polynomials, International Journal of Advanced Scientific and Technical Research, Vol.4 Issue 2 August 2012.