International Journal of Engineering Research and Development
e-ISSN: 2278-067X, p-ISSN: 2278-800X, www.ijerd.com
Volume 9, Issue 2 (November 2013), PP. 41-52

Location of Zero-free Regions of Polynomials

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Abstract: In this paper we locate zero-free regions of polynomials when their coefficients are restricted to certain conditions.
Mathematics Subject Classification: 30C10, 30C15

Keywords and phrases: Coefficient, Polynomial, Zero.

I. INTRODUCTION AND STATEMENT OF RESULTS

The problem of finding the regions containing all or some or no zero of a polynomial is very important in the theory of polynomials. In this connection, a lot of papers have been published by various researchers (e.g. see 1,2,3,4,5,6). Recently, M. H. Gulzar[5] proved the following results:

Theorem A: Let Let $P(z)=\sum_{j=0}^{\infty} a_j z^j$ be a polynomial o f degree n such that

$\operatorname{Re}(a_j)=\alpha_j$, $\operatorname{Im}(a_j)=\beta_j$ and for some positive integers λ, μ and for some real numbers

$k_1, k_2, \tau_1, \tau_2; 0<k_1\le 1, 0<k_2\le 1, 0<\tau_1\le 1, 0<\tau_2\le 1,$

$$k_1\alpha_n \le \alpha_{n-1} \le .....\le \alpha_{\lambda+1} \le \alpha_\lambda \ge \alpha_{\lambda-1} \ge .....\ge \alpha_1 \ge \tau_1\alpha_0$$

$$k_2\beta_n \le \beta_{n-1} \le .....\le \beta_{\mu+1} \le \beta_\mu \ge \beta_{\mu-1} \ge .....\ge \beta_1 \ge \tau_1\beta_0 .$$

Then the number of zeros of P(z) in $|z|\le \frac{R}{c}(R>0, c>1)$ does not exceed $\frac{1}{\log c}\log\frac{M_1}{|a_0|}$, where

$$M_1=|a_n|R^{n+1}+|a_0|+R^n[|\alpha_n|+|\beta_n|-k_1(|\alpha_n|+\alpha_n)-k_2(|\beta_n|+\beta_n)+\alpha_\lambda+\beta_\mu]$$
$$+R^\lambda[\alpha_\lambda-\tau_1(|\alpha_0|+\alpha_0)+|\alpha_0|]+R^\mu[\beta_\mu-\tau_2(|\beta_0|+\beta_0)+|\beta_0|] \text{ for } R\ge 1$$

and

$$M_1=|a_n|R^{n+1}+|a_0|+R^n[|\alpha_n|+|\beta_n|-k_1(|\alpha_n|+\alpha_n)-k_2(|\beta_n|+\beta_n)+\alpha_\lambda+\beta_\mu]$$
$$+R[\alpha_\lambda+\beta_\mu-\tau_1(|\alpha_0|+\alpha_0)-\tau_2(|\beta_0|+\beta_0)+|\alpha_0|+|\beta_0|] \text{ for } R\le 1.$$

Theorem B: Let Let $P(z)=\sum_{j=0}^{\infty} a_j z^j$ be a polynomial o f degree n such that

$\operatorname{Re}(a_j)=\alpha_j$, $\operatorname{Im}(a_j)=\beta_j$ and for some positive integers $\lambda, \mu, \rho, \sigma$ and for some real numbers

$k_1, k_2, k_3, k_4; \tau_1, \tau_2, \tau_3, \tau_4; 0<k_1\le 1, 0<k_2\le 1, 0<k_3\le 1, 0<k_4\le 1$

$0<\tau_1\le 1, 0<\tau_2\le 1, 0<\tau_3\le 1, 0<\tau_4\le 1$,

$$k_1\alpha_{2[\frac{n}{2}]} \le .....\le \alpha_{2\lambda+2} \le \alpha_{2\lambda} \ge .....\ge \alpha_2 \ge \tau_1\alpha_0$$

$$k_2\alpha_{2[\frac{n}{2}]-1} \le .....\le \alpha_{2\mu+1} \le \alpha_{2\mu-1} \ge .....\ge \alpha_3 \ge \tau_2\alpha_1$$

$$k_3\beta_{2[\frac{n}{2}]} \le .....\le \beta_{2\rho+2} \le \beta_{2\rho} \ge .....\ge \beta_2 \ge \tau_3\beta_0$$

$$k_2\beta_{2[\frac{n}{2}]-1} \le .....\le \beta_{2\sigma+1} \le \beta_{2\sigma-1} \ge .....\ge \beta_3 \ge \tau_4\beta_1 .$$

.

Then, for even n, the number of zeros of P(z) in in $|z| \le \frac{R}{c}(R > 0, c > 1)$ does not exceed $\frac{1}{\log c}\log\frac{M_2}{|a_0|}$,

where

$$M_2 = |a_n|R^{n+1} + |a_0| + R^n[|\alpha_n| + |\beta_n| - k_1(|\alpha_n| + \alpha_n) - k_2(|\beta_n| + \beta_n) + \alpha_\lambda + \beta_\mu] + R^\lambda[\alpha_\lambda - \tau_1(|\alpha_0| + \alpha_0) + |\alpha_0|] + R^\mu[\beta_\mu - \tau_2(|\beta_0| + \beta_0) + |\beta_0|] \text{ for } R \ge 1$$

and

$$M_2 = |a_n|R^{n+1} + |a_0| + R^n[|\alpha_n| + |\beta_n| - k_1(|\alpha_n| + \alpha_n) - k_2(|\beta_n| + \beta_n) + \alpha_\lambda + \beta_\mu] + R[\alpha_\lambda + \beta_\mu - \tau_1(|\alpha_0| + \alpha_0) - \tau_2(|\beta_0| + \beta_0) + |\alpha_0| + |\beta_0|] \text{ for } R \le 1.$$

If n is odd, the number of zeros of P(z) in $|z| \le \frac{R}{c}(R > 0, c > 1)$ does not exceed $\frac{1}{\log c}\log\frac{M_2'}{|a_0|}$, where

M_2' is same as M_2 except that k_1, k_2, k_3, k_4 are respectively replaced by k_2, k_1, k_4, k_3 and $\tau_1, \tau_2, \tau_3, \tau_4$ are respectively replaced by $\tau_2, \tau_1, \tau_4, \tau_3$.

Theorem C: Let Let $P(z) = \sum_{j=0}^{\infty} a_j z^j$ be a polynomial o f degree n such that

for some positive integers $\lambda, \mu,$ and for some real numbers $k_1, k_2, \tau_1, \tau_2,$
$0 < k_1 \le 1, 0 < k_2 \le 1, 0 < \tau_1 \le 1, 0 < \tau_2 \le 1,$

$$k_1\left|a_{2[\frac{n}{2}]}\right| \le \ldots\ldots \le |a_{2\lambda+2}| \le |a_{2\lambda}| \ge \ldots\ldots \ge |a_2| \ge \tau_1|a_0|$$

$$k_2\left|a_{2[\frac{n}{2}]-1}\right| \le \ldots\ldots \le |a_{2\mu+1}| \le |a_{2\mu-1}| \ge \ldots\ldots \ge |a_3| \ge \tau_2|a_1|.$$

Then for even n the number of zeros of P(z) in $|z| \le \frac{R}{c}(R > 0, c > 1)$ does not exceed $\frac{1}{\log c}\log\frac{M_3}{|a_0|}$,

where for $R \ge 1$,

$$M_3 = |a_n|R^{n+2} + |a_{n-1}|R^{n+1} + |a_0| + R^n[|a_1| + (1-k_1)|a_n| + (1-k_2)|a_{n-1}| + \cos\alpha(2|a_{2\lambda}| + 2|a_{2\mu}| - k_1|a_n| - k_2|a_{n-1}| - \tau_2|a_1| - \tau_1|a_0|) + \sin\alpha(k_1|a_n| + k_2|a_{n-1}| + \tau_2|a_1| + \tau_1|a_0| + 2\sum_{j=2}^{n-2}|a_j|)]$$

and for $R \le 1$,

$$M_3 = |a_n|R^{n+2} + |a_{n-1}|R^{n+1} + |a_0| + R[|a_1| + (1-k_1)|a_n| + (1-k_2)|a_{n-1}| + \cos\alpha(2|a_{2\lambda}| + 2|a_{2\mu}| - k_1|a_n| - k_2|a_{n-1}| - \tau_2|a_1| - \tau_1|a_0|) + \sin\alpha(k_1|a_n| + k_2|a_{n-1}| + \tau_2|a_1| + \tau_1|a_0| + 2\sum_{j=2}^{n-2}|a_j|)].$$

If n is odd, then the number of zeros of P(z) in $|z| \le \frac{R}{c}(R > 0, c > 1)$ does not exceed $\frac{1}{\log c}\log\frac{M_4}{|a_0|}$,

where M_4 is same as M_3 except that k_1, k_2 are respectively replaced by k_2, k_1 and τ_1, τ_2 are respectively replaced by τ_2, τ_1.

In this paper we find regions containing no zero of the polynomials in theorems 1,2 3 and prove the following results:

Theorem 1: Let Let $P(z)=\sum_{j=0}^{\infty} a_j z^j$ be a polynomial o f degree n such that

$\mathrm{Re}(a_j)=\alpha_j$, $\mathrm{Im}(a_j)=\beta_j$ and for some positive integers λ,μ and for some real numbers $k_1,k_2,\tau_1,\tau_2; 0<k_1\le 1, 0<k_2\le 1, 0<\tau_1\le 1, 0<\tau_2\le 1,$

$$k_1\alpha_n \le \alpha_{n-1} \le \ldots\ldots \le \alpha_{\lambda+1} \le \alpha_\lambda \ge \alpha_{\lambda-1} \ge \ldots\ldots \ge \alpha_1 \ge \tau_1\alpha_0$$

$$k_2\beta_n \le \beta_{n-1} \le \ldots\ldots \le \beta_{\mu+1} \le \beta_\mu \ge \beta_{\mu-1} \ge \ldots\ldots \ge \beta_1 \ge \tau_1\beta_0.$$

Then that P(z) has no zero in $|z|<\frac{|a_0|}{M^*}$, for $R\ge 1$ and no zero in $|z|<\frac{|a_0|}{M^{**}}$ for $R\le 1$, where

$$M^* = |a_n|R^{n+1}+R^n[|\alpha_n|+|\beta_n|-k_1(|\alpha_n|+\alpha_n)-k_2(|\beta_n|+\beta_n)+\alpha_\lambda+\beta_\mu]$$
$$+R^\lambda[\alpha_\lambda-\tau_1(|\alpha_0|+\alpha_0)+|\alpha_0|]+R^\mu[\beta_\mu-\tau_2(|\beta_0|+\beta_0)+|\beta_0|]$$

and

$$M^{**} = |a_n|R^{n+1}+R^n[|\alpha_n|+|\beta_n|-k_1(|\alpha_n|+\alpha_n)-k_2(|\beta_n|+\beta_n)+\alpha_\lambda+\beta_\mu]$$
$$+R[\alpha_\lambda+\beta_\mu-\tau_1(|\alpha_0|+\alpha_0)-\tau_2(|\beta_0|+\beta_0)+|\alpha_0|+|\beta_0|].$$

Combining Theorem 1 and Theorem A, we get the following result:

Corollary 1: Let Let $P(z)=\sum_{j=0}^{\infty} a_j z^j$ be a polynomial o f degree n such that

$\mathrm{Re}(a_j)=\alpha_j$, $\mathrm{Im}(a_j)=\beta_j$ and for some positive integers λ,μ and for some real numbers $k_1,k_2,\tau_1,\tau_2; 0<k_1\le 1, 0<k_2\le 1, 0<\tau_1\le 1, 0<\tau_2\le 1,$

$$k_1\alpha_n \le \alpha_{n-1} \le \ldots\ldots \le \alpha_{\lambda+1} \le \alpha_\lambda \ge \alpha_{\lambda-1} \ge \ldots\ldots \ge \alpha_1 \ge \tau_1\alpha_0$$

$$k_2\beta_n \le \beta_{n-1} \le \ldots\ldots \le \beta_{\mu+1} \le \beta_\mu \ge \beta_{\mu-1} \ge \ldots\ldots \ge \beta_1 \ge \tau_1\beta_0.$$

Then the number of zeros of P(z) in $\frac{|a_0|}{M^*}\le |z| \le \frac{R}{c}(R>0,c>1)$ does not exceed $\frac{1}{\log c}\log\frac{M_1}{|a_0|}$ for $R\ge 1$ and the number of zeros of P(z) in $\frac{|a_0|}{M^{**}}\le |z| \le \frac{R}{c}(R>0,c>1)$ does not exceed $\frac{1}{\log c}\log\frac{M_1}{|a_0|}$ for $R\le 1$, where M_1, M^*, M^{**} are given as in Theorem 1 and Theorem A.

Theorem 2: Let Let $P(z)=\sum_{j=0}^{\infty} a_j z^j$ be a polynomial o f degree n such that

$\mathrm{Re}(a_j)=\alpha_j$, $\mathrm{Im}(a_j)=\beta_j$ and for some positive integers λ,μ,ρ,σ and for some real numbers $k_1,k_2,k_3,k_4;\tau_1,\tau_2,\tau_3,\tau_4; 0<k_1\le 1, 0<k_2\le 1, 0<k_3\le 1, 0<k_4\le 1$
$0<\tau_1\le 1, 0<\tau_2\le 1, 0<\tau_3\le 1, 0<\tau_4\le 1,$

$$k_1\alpha_{2[\frac{n}{2}]} \le \ldots\ldots \le \alpha_{2\lambda+2} \le \alpha_{2\lambda} \ge \ldots\ldots \ge \alpha_2 \ge \tau_1\alpha_0$$

$$k_2\alpha_{2[\frac{n}{2}]-1} \le \ldots\ldots \le \alpha_{2\mu+1} \le \alpha_{2\mu-1} \ge \ldots\ldots \ge \alpha_3 \ge \tau_2\alpha_1$$

$$k_3\beta_{2[\frac{n}{2}]} \le \ldots\ldots \le \beta_{2\rho+2} \le \beta_{2\rho} \ge \ldots\ldots \ge \beta_2 \ge \tau_3\beta_0$$

$$k_2\beta_{2[\frac{n}{2}]-1} \le \ldots\ldots \le \beta_{2\sigma+1} \le \beta_{2\sigma-1} \ge \ldots\ldots \ge \beta_3 \ge \tau_4\beta_1.$$

.

Then, for even n, P(z) has no zero in $|z| < \frac{|a_0|}{M_1^*}$, for $R \geq 1$ and no zero in $|z| < \frac{|a_0|}{M_1^{**}}$ for $R \leq 1$, where

$$\begin{aligned} M_1^* = {} & |a_n| R^{n+2} + |a_{n-1}| R^{n+1} + R^n[|\alpha_n| + |\beta_n| + |\alpha_{n-1}| + |\beta_{n-1}| \\ & + 2(\alpha_{2\lambda} + \alpha_{2\mu-1} + \beta_{2\rho} + \beta_{2\sigma-1}) - (k_1|\alpha_n| + k_3|\beta_n|) \\ & - (k_2|\alpha_{n-1}| + k_4|\beta_{n-1}|) - (\tau_2\alpha_1 + \tau_4\beta_1) - (\tau_1\alpha_0 + \tau_3\beta_0) \\ & + (1-\tau_2)|\alpha_1| + (1-\tau_1)|\alpha_0| + (1-\tau_3)|\beta_1|(1-\tau_4)|\beta_0| + |a_1|] \end{aligned}$$

and

$$\begin{aligned} M_1^{**} = {} & |a_n| R^{n+2} + |a_{n-1}| R^{n+1} + R[|\alpha_n| + |\beta_n| + |\alpha_{n-1}| + |\beta_{n-1}| \\ & + 2(\alpha_{2\lambda} + \alpha_{2\mu-1} + \beta_{2\rho} + \beta_{2\sigma-1}) - (k_1|\alpha_n| + k_3|\beta_n|) \\ & - (k_2|\alpha_{n-1}| + k_4|\beta_{n-1}|) - (\tau_2\alpha_1 + \tau_4\beta_1) - (\tau_1\alpha_0 + \tau_3\beta_0) \\ & + (1-\tau_2)|\alpha_1| + (1-\tau_1)|\alpha_0| + (1-\tau_3)|\beta_1|(1-\tau_4)|\beta_0| + |a_1|]. \end{aligned}$$

If n is odd, then P(z) has no zero in $|z| < \frac{|a_0|}{M_2^*}$, for $R \geq 1$ and no zero in $|z| < \frac{|a_0|}{M_2^{**}}$ for $R \leq 1$, where M_2^* and M_2^{**} are same as M_1^* and M_1^{**} except that k_1, k_2, k_3, k_4 are respectively replaced by k_2, k_1, k_4, k_3 and $\tau_1, \tau_2, \tau_3, \tau_4$ are respectively replaced by $\tau_2, \tau_1, \tau_4, \tau_3$.

Combining Theorem 2 and Theorem B, we get the following result:

Corollary 2: Let Let $P(z) = \sum_{j=0}^{\infty} a_j z^j$ be a polynomial o f degree n such that

$\mathrm{Re}(a_j) = \alpha_j$, $\mathrm{Im}(a_j) = \beta_j$ and for some positive integers $\lambda, \mu, \rho, \sigma$ and for some real numbers $k_1, k_2, k_3, k_4; \tau_1, \tau_2, \tau_3, \tau_4; 0 < k_1 \leq 1, 0 < k_2 \leq 1, 0 < k_3 \leq 1, 0 < k_4 \leq 1$ $0 < \tau_1 \leq 1, 0 < \tau_2 \leq 1, 0 < \tau_3 \leq 1, 0 < \tau_4 \leq 1$,

$$k_1\alpha_{2[\frac{n}{2}]} \leq \ldots\ldots \leq \alpha_{2\lambda+2} \leq \alpha_{2\lambda} \geq \ldots\ldots \geq \alpha_2 \geq \tau_1\alpha_0$$

$$k_2\alpha_{2[\frac{n}{2}]-1} \leq \ldots\ldots \leq \alpha_{2\mu+1} \leq \alpha_{2\mu-1} \geq \ldots\ldots \geq \alpha_3 \geq \tau_2\alpha_1$$

$$k_3\beta_{2[\frac{n}{2}]} \leq \ldots\ldots \leq \beta_{2\rho+2} \leq \beta_{2\rho} \geq \ldots\ldots \geq \beta_2 \geq \tau_3\beta_0$$

$$k_2\beta_{2[\frac{n}{2}]-1} \leq \ldots\ldots \leq \beta_{2\sigma+1} \leq \beta_{2\sigma-1} \geq \ldots\ldots \geq \beta_3 \geq \tau_4\beta_1.$$

.

Then, for even n, the number of zeros of P(z) in $\frac{|a_0|}{M_3^*} \leq |z| \leq \frac{R}{c} (R > 0, c > 1)$ does not exceed $\frac{1}{\log c} \log \frac{M_2}{|a_0|}$ for $R \geq 1$ and the number of zeros of P(z) in $\frac{|a_0|}{M_3^{**}} \leq |z| \leq \frac{R}{c} (R > 0, c > 1)$ does not exceed $\frac{1}{\log c} \log \frac{M_2}{|a_0|}$ for $R \leq 1$, where

M_2, M_3^*, M_3^{**} are as given in Theorem 2 and Theorem B.

If n is odd, the number of zeros of P(z) in $\frac{|a_0|}{M_4^{\ *}} \le |z| \le \frac{R}{c}(R>0, c>1)$ does not exceed $\frac{1}{\log c}\log\frac{M_2^{\ \prime}}{|a_0|}$ for $R \ge 1$ and the number of zeros of P(z) in $\frac{|a_0|}{M_4^{\ **}} \le |z| \le \frac{R}{c}(R>0, c>1)$ does not exceed $\frac{1}{\log c}\log\frac{M_2^{\ \prime}}{|a_0|}$, where $M_2^{\ \prime}, M_4^{\ *}, M_4^{\ **}$ are same as M_2, $M_3^{\ *}, M_3^{\ **}$ except that k_1, k_2, k_3, k_4 are respectively replaced by k_2, k_1, k_4, k_3 and $\tau_1, \tau_2, \tau_3, \tau_4$ are respectively replaced by $\tau_2, \tau_1, \tau_4, \tau_3$.

Theorem 3: Let Let $P(z)=\sum_{j=0}^{\infty} a_j z^j$ be a polynomial o f degree n such that

for some positive integers $\lambda, \mu,$ and for some real numbers $k_1, k_2, \tau_1, \tau_2,$
$0<k_1 \le 1, 0<k_2 \le 1, 0<\tau_1 \le 1, 0<\tau_2 \le 1,$

$$k_1\left|a_{2[\frac{n}{2}]}\right| \le \ldots\ldots \le |a_{2\lambda+2}| \le |a_{2\lambda}| \ge \ldots\ldots \ge |a_2| \ge \tau_1 |a_0|$$

$$k_2\left|a_{2[\frac{n}{2}]-1}\right| \le \ldots\ldots \le |a_{2\mu+1}| \le |a_{2\mu-1}| \ge \ldots\ldots \ge |a_3| \ge \tau_2 |a_1|.$$

Then for even n P(z) has no zero in $|z| < \frac{|a_0|}{M_5^{\ *}}$, for $R \ge 1$ and no zero in $|z| < \frac{|a_0|}{M_5^{\ **}}$ for $R \le 1$, where

$$\begin{aligned} M_5^{\ *} = {} & |a_n|R^{n+2} + |a_{n-1}|R^{n+1} + R^n[|a_1| + (1-k_1)|a_n| + (1-k_2)|a_{n-1}| \\ & + \cos\alpha(2|a_{2\lambda}| + 2|a_{2\mu}| - k_1|a_n| - k_2|a_{n-1}| - \tau_2|a_1| - \tau_1|a_0|) \\ & + \sin\alpha(k_1|a_n| + k_2|a_{n-1}| + \tau_2|a_1| + \tau_1|a_0| + 2\sum_{j=2}^{n-2}|a_j|)] \end{aligned}$$

and

$$\begin{aligned} M_5^{\ **} = {} & |a_n|R^{n+2} + |a_{n-1}|R^{n+1} + R[|a_1| + (1-k_1)|a_n| + (1-k_2)|a_{n-1}| \\ & + \cos\alpha(2|a_{2\lambda}| + 2|a_{2\mu}| - k_1|a_n| - k_2|a_{n-1}| - \tau_2|a_1| - \tau_1|a_0|) \\ & + \sin\alpha(k_1|a_n| + k_2|a_{n-1}| + \tau_2|a_1| + \tau_1|a_0| + 2\sum_{j=2}^{n-2}|a_j|)] \end{aligned}$$

and for odd n P(z) has no zero in $|z| < \frac{|a_0|}{M_6^{\ *}}$, for $R \ge 1$ and no zero in $|z| < \frac{|a_0|}{M_6^{\ **}}$ for $R \le 1$, where $M_6^{\ *}$ and $M_6^{\ **}$ are same as $M_5^{\ *}$ and $M_5^{\ **}$ except that k_1, k_2 are respectively replaced by k_2, k_1 and τ_1, τ_2 are respectively replaced by τ_2, τ_1.

Combining Theorem 3 and Theorem C, we get the following result:

Corollary 3: Let Let $P(z)=\sum_{j=0}^{\infty} a_j z^j$ be a polynomial o f degree n such that

for some positive integers $\lambda, \mu,$ and for some real numbers $k_1, k_2, \tau_1, \tau_2,$
$0<k_1 \le 1, 0<k_2 \le 1, 0<\tau_1 \le 1, 0<\tau_2 \le 1,$

$$k_1\left|a_{2[\frac{n}{2}]}\right| \le \ldots\ldots \le \left|a_{2\lambda+2}\right| \le \left|a_{2\lambda}\right| \ge \ldots\ldots \ge \left|a_2\right| \ge \tau_1\left|a_0\right|$$

$$k_2\left|a_{2[\frac{n}{2}]-1}\right| \le \ldots\ldots \le \left|a_{2\mu+1}\right| \le \left|a_{2\mu-1}\right| \ge \ldots\ldots \ge \left|a_3\right| \ge \tau_2\left|a_1\right|.$$

Then for even n the number of zeros of P(z) in $\frac{|a_0|}{M_5^{*}} \le |z| \le \frac{R}{c}(R>0, c>1)$ does not exceed $\frac{1}{\log c}\log\frac{M_3}{|a_0|}$ for $R \ge 1$ and the number of zeros of P(z) in $\frac{|a_0|}{M_5^{**}} \le |z| \le \frac{R}{c}(R>0, c>1)$ does not exceed $\frac{1}{\log c}\log\frac{M_3}{|a_0|}$ for $R \le 1$,where M_3, M_5^{*}, M_5^{**} are as given in Theorem 3 and Theorem C.

If n is odd, then the number of zeros of P(z) in $\frac{|a_0|}{M_6^{*}} \le |z| \le \frac{R}{c}(R>0, c>1)$ does not exceed $\frac{1}{\log c}\log\frac{M_4}{|a_0|}$ for $R \ge 1$ and the number of zeros of P(z) in $\frac{|a_0|}{M_4^{**}} \le |z| \le \frac{R}{c}(R>0, c>1)$ does not exceed $\frac{1}{\log c}\log\frac{M_4}{|a_0|}$ for $R \le 1$,where M_4, M_6^{*}, M_6^{**} are same as M_3, M_5^{*}, M_5^{**} except that k_1, k_2 are respectively replaced by k_2, k_1 and τ_1, τ_2 are respectively replaced by τ_2, τ_1.

For different values of the parameters, we get many interesting results from the above theorems.

II. PROOFS OF THEOREMS

Proof of Theorem 1: Consider the polynomial

$$\begin{aligned}
F(z) &= (1-z)P(z) \\
&= (1-z)(a_n z^n + a_{n-1}z^{n-1} + \ldots\ldots + a_1 z + a_0) \\
&= -a_n z^{n+1} + (a_n - a_{n-1})z^n + \ldots\ldots + (a_1 - a_0)z + a_0 \\
&= -a_n z^{n+1} + a_0 - (k_1-1)\alpha_n z^n + (k_1\alpha_n - \alpha_{n-1})z^n + (\alpha_{n-1} - \alpha_{n-2})z^{n-1} \\
&\quad + \ldots\ldots + (\alpha_{\lambda+1} - \alpha_\lambda)z^{\lambda+1} + (\alpha_\lambda - \alpha_{\lambda-1})z^\lambda + \ldots\ldots + (\alpha_1 - \tau_1\alpha_0)z \\
&\quad + (\tau_1 - 1)\alpha_0 z + i\{-(k_2-1)\beta_n z^n + (k_2\beta_n - \beta_{n-1})z^n + (\beta_{n-1} - \beta_{n-2})z^{n-1} + \ldots\ldots \\
&\quad + (\beta_{\mu+1} - \beta_\mu)z^{\mu+1} + (\beta_\mu - \beta_{\mu-1})z^\mu + \ldots\ldots + (\beta_1 - \tau_2\beta_0)z + (\tau_2 - 1)\beta_0 z\} \\
&= a_0 + G(z) \text{ ,where}
\end{aligned}$$

$$\begin{aligned}
G(z) &= -a_n z^{n+1} - (k_1-1)\alpha_n z^n + (k_1\alpha_n - \alpha_{n-1})z^n + (\alpha_{n-1} - \alpha_{n-2})z^{n-1} \\
&\quad + \ldots\ldots + (\alpha_{\lambda+1} - \alpha_\lambda)z^{\lambda+1} + (\alpha_\lambda - \alpha_{\lambda-1})z^\lambda + \ldots\ldots + (\alpha_1 - \tau_1\alpha_0)z \\
&\quad + (\tau_1 - 1)\alpha_0 z + i\{-(k_2-1)\beta_n z^n + (k_2\beta_n - \beta_{n-1})z^n + (\beta_{n-1} - \beta_{n-2})z^{n-1} + \ldots\ldots \\
&\quad + (\beta_{\mu+1} - \beta_\mu)z^{\mu+1} + (\beta_\mu - \beta_{\mu-1})z^\mu + \ldots\ldots + (\beta_1 - \tau_2\beta_0)z + (\tau_2 - 1)\beta_0 z\}
\end{aligned}$$

For $|z| \le R$, we have by using the hypothesis

$$|G(z)| \le |a_n|R^{n+1} + (1-k_1)|\alpha_n|R^n + (\alpha_{n-1} - k_1\alpha_n)R^n + (\alpha_{n-2} - \alpha_{n-1})R^{n-1} + \ldots\ldots$$

$$+(\alpha_\lambda-\alpha_{\lambda+1})R^{\lambda+1}+(\alpha_\lambda-\alpha_{\lambda-1})R^{\lambda}+\ldots\ldots+(\alpha_1-\tau_1\alpha_0)R+(1-\tau_1)|\alpha_0|R$$
$$+(1-k_2)|\beta_n|R^n+(\beta_{n-1}-k_2\beta_n)R^n+(\beta_{n-2}-\beta_{n-1})R^{n-1}+\ldots\ldots$$
$$+(\beta_\mu-\beta_{\mu+1})R^{\mu+1}+(\beta_\mu-\beta_{\mu-1})R^{\mu}+\ldots\ldots+(\beta_1-\tau_2\beta_0)R+(1-\tau_2)|\beta_0|R$$

For $R\ge 1$,

$$|G(z)|\le|a_n|R^{n+1}+R^n[(1-k_1)|\alpha_n|+\alpha_{n-1}-k_1\alpha_n+\alpha_{n-2}-\alpha_{n-1}+\ldots\ldots$$
$$+\alpha_{\lambda+2}-\alpha_{\lambda+1}+\alpha_\lambda-\alpha_{\lambda+1}+(1-k_2)|\beta_n|+\beta_{n-1}-k_2\beta_n+\beta_{n-2}-\beta_{n-1}+\ldots\ldots$$
$$+\beta_{\mu+2}-\beta_{\mu+1}+\beta_\mu-\beta_{\mu+1}]+R^{\lambda}[\alpha_\lambda-\alpha_{\lambda-1}+\alpha_{\lambda-1}-\alpha_{\lambda-2}+\ldots\ldots$$
$$+\alpha_1-\tau_1\alpha_0+(1-\tau_1)|\alpha_0|]+R^{\mu}[\beta_\mu-\beta_{\mu-1}+\beta_{\mu-1}-\beta_{\mu-2}+\ldots\ldots$$
$$+\beta_1-\tau_2\beta_0+(1-\tau_2)|\beta_0|]$$
$$=|a_n|R^{n+1}+R^n[|\alpha_n|+|\beta_n|-k_1(|\alpha_n|+\alpha_n)-k_2(|\beta_n|+\beta_n)+\alpha_\lambda+\beta_\mu]$$
$$+R^{\lambda}[\alpha_\lambda-\tau_1(|\alpha_0|+\alpha_0)+|\alpha_0|]+R^{\mu}[\beta_\mu-\tau_2(|\beta_0|+\beta_0)+|\beta_0|]$$
$$=M^*$$

For $R\le 1$,

$$|G(z)|\le|a_n|R^{n+1}+R^n[|\alpha_n|+|\beta_n|-k_1(|\alpha_n|+\alpha_n)-k_2(|\beta_n|+\beta_n)+\alpha_\lambda+\beta_\mu]$$
$$+R[\alpha_\lambda+\beta_\mu-\tau_1(|\alpha_0|+\alpha_0)-\tau_2(|\beta_0|+\beta_0)+|\alpha_0|+|\beta_0|]$$
$$=M^{**}$$

Since G(z) is analytic for $|z|\le R$ b and G(0)=0, it follows by Schwarz Lemma that

$|G(z)|\le M^*|z|$ for $R\ge 1$ and $|G(z)|\le M^{**}|z|$ for $R\le 1$.

Hence, for $R\ge 1$,

$$|F(z)|=|a_0+G(z)|$$
$$\ge|a_0|-|G(z)|$$
$$\ge|a_0|-M^*|z|$$
$$>0$$

if $|z|<\dfrac{|a_0|}{M^*}$.

And for $R\le 1$,

$$|F(z)|=|a_0+G(z)|$$
$$\ge|a_0|-|G(z)|$$
$$\ge|a_0|-M^{**}|z|$$
$$>0$$

if $|z|<\dfrac{|a_0|}{M^{**}}$.

This shows that F(z) has no zero in $|z|<\dfrac{|a_0|}{M^*}$, for $R\ge 1$ and no zero in $|z|<\dfrac{|a_0|}{M^{**}}$ for $R\le 1$.

Since the zeros of P(z) are also the zeros of F(z) , it follows that that P(z) has no zero in $|z| < \frac{|a_0|}{M^*}$, for $R \geq 1$ and no zero in $|z| < \frac{|a_0|}{M^{**}}$ for $R \leq 1$, thereby proving Theorem 1.

Proof of Theorem 2: Let n be even. Consider the polynomial

$$\begin{aligned}
F(z) &= (1-z^2)P(z) = (1-z^2)(a_n z^n + a_{n-1}z^{n-1} + \ldots\ldots + a_1 z + a_0) \\
&= -az^{n+2} - a_{n-1}z^{n+1} + (a_n - a_{n-2})z^n + (a_{n-1} - a_{n-3})z^{n-1} + \ldots\ldots + (a_3 - a_1)z^3 \\
&\quad + (a_2 - a_0)z^2 + a_1 z + a_0 \\
&= -a_n z^{n+2} - a_{n-1}z^{n+1} - (k_1 - 1)\alpha_n z^n + (k_1\alpha_n - \alpha_{n-2})z^n - (k_2 - 1)z^{n-1} \\
&\quad + (k_2\alpha_{n-1} - \alpha_{n-3})z^{n-1} + (\alpha_{n-2} - \alpha_{n-4})z^{n-2} + (\alpha_{n-3} - \alpha_{n-5})z^{n-3} + \ldots\ldots \\
&\quad + (\alpha_3 - \tau_2\alpha_1)z^3 + (\tau_2\alpha_1 - \alpha_1)z^3 + (\alpha_2 - \tau_1\alpha_0)z^2 + (\tau_1\alpha_0 - \alpha_0)z^2 \\
&\quad + a_1 z + a_0 + i\{-(k_3 - 1)\beta_n z^n + (k_3\beta_n - \beta_{n-2})z^n - (k_4 - 1)\beta_{n-1}z^{n-1} \\
&\quad + (k_4\beta_{n-1} - \beta_{n-3})z^{n-1} + (\beta_{n-2} - \beta_{n-4})z^{n-2} + (\beta_{n-3} - \beta_{n-5})z^{n-3} + \ldots\ldots \\
&\quad + (\beta_3 - \tau_4\beta_1)z^3 + (\tau_4\beta_1 - \beta_1)z^3 + (\beta_2 - \tau_3\beta_0)z^2 + (\tau_3\beta_0 - \beta_0)z^2. \\
&= a_0 + G(z)
\end{aligned}$$

, where

$$\begin{aligned}
G(z) &= -a_n z^{n+2} - a_{n-1}z^{n+1} - (k_1 - 1)\alpha_n z^n + (k_1\alpha_n - \alpha_{n-2})z^n - (k_2 - 1)z^{n-1} \\
&\quad + (k_2\alpha_{n-1} - \alpha_{n-3})z^{n-1} + (\alpha_{n-2} - \alpha_{n-4})z^{n-2} + (\alpha_{n-3} - \alpha_{n-5})z^{n-3} + \ldots\ldots \\
&\quad + (\alpha_3 - \tau_2\alpha_1)z^3 + (\tau_2\alpha_1 - \alpha_1)z^3 + (\alpha_2 - \tau_1\alpha_0)z^2 + (\tau_1\alpha_0 - \alpha_0)z^2 \\
&\quad + a_1 z + i\{-(k_3 - 1)\beta_n z^n + (k_3\beta_n - \beta_{n-2})z^n - (k_4 - 1)\beta_{n-1}z^{n-1} \\
&\quad + (k_4\beta_{n-1} - \beta_{n-3})z^{n-1} + (\beta_{n-2} - \beta_{n-4})z^{n-2} + (\beta_{n-3} - \beta_{n-5})z^{n-3} + \ldots\ldots \\
&\quad + (\beta_3 - \tau_4\beta_1)z^3 + (\tau_4\beta_1 - \beta_1)z^3 + (\beta_2 - \tau_3\beta_0)z^2 + (\tau_3\beta_0 - \beta_0)z^2.
\end{aligned}$$

For $|z| \leq R$, we have by using the hypothesis

$$\begin{aligned}
|G(z)| &\leq |a_n|R^{n+2} + |a_{n-1}|R^{n+1} + (1-k_1)|\alpha_n|R^n + (\alpha_{n-2} - k_1\alpha_n)R^n + (1-k_2)|\alpha_{n-1}|R^{n-1} \\
&\quad + (\alpha_{n-3} - k_2\alpha_{n-1})R^{n-1} + (\alpha_{n-4} - \alpha_{n-2})R^{n-2} + (\alpha_{n-5} - \alpha_{n-3})R^{n-3} + \ldots\ldots \\
&\quad + (\alpha_{2\lambda} - \alpha_{2\lambda+2})R^{2\lambda+2} + (\alpha_{2\lambda} - \alpha_{2\lambda-2})R^{2\lambda} \ldots\ldots + (\alpha_{2\mu-1} - \alpha_{2\mu+1})R^{2\mu+1} \\
&\quad + (\alpha_{2\mu-1} - \alpha_{2\mu-3})R^{2\mu-1} + \ldots\ldots + (\alpha_3 - \tau_2\alpha_1)R^3 + (1-\tau_2)|\alpha_1|R^3 \\
&\quad + (\alpha_2 - \tau_1\alpha_0)R^2 + (1-\tau_1)|\alpha_0|R^2 + |a_1|R + (1-k_3)|\beta_n|R^n \\
&\quad + (\beta_{n-2} - k_3\beta_n)R^n + \ldots\ldots + (\beta_{2\rho} - \beta_{2\rho+2})R^{2\rho+2} + (\beta_{2\rho} - \beta_{2\rho-2})R^{2\rho} \\
&\quad + \ldots\ldots + (\beta_{2\sigma-1} - \beta_{2\sigma+1})R^{2\sigma+1} + (\beta_{2\sigma-1} - \beta_{2\sigma-3})R^{2\sigma-1} + (\beta_3 - \tau_4\beta_1)R^3 \\
&\quad + (1-\tau_4)|\beta_1|R^3 + (\beta_2 - \tau_3\beta_0)R^2 + (1-\tau_3)|\beta_0|R^2
\end{aligned}$$

For $R \geq 1$,

$$|G(z)| \leq |a_n|R^{n+2} + |a_{n-1}|R^{n+1} + R^n[(1-k_1)|\alpha_n| + \alpha_{n-2} - k_1\alpha_n + (1-k_2)|\alpha_{n-1}|$$

$$+\alpha_{n-3}-k_2\alpha_{n-1}+\alpha_{n-4}-\alpha_{n-2}+\alpha_{n-5}-\alpha_{n-3}+\ldots\ldots$$
$$+\alpha_{2\lambda}-\alpha_{2\lambda+2}+\alpha_{2\lambda}-\alpha_{2\lambda-2}\ldots\ldots+\alpha_{2\mu-1}-\alpha_{2\mu+1}$$
$$+\alpha_{2\mu-1}-\alpha_{2\mu-3}+\ldots\ldots+\alpha_3-\tau_2\alpha_1+(1-\tau_2)|\alpha_1|$$
$$+(\alpha_2-\tau_1\alpha_0)+(1-\tau_1)|\alpha_0|+|a_0|+(1-k_3)|\beta_n|$$
$$+\beta_{n-2}-k_3\beta_n+\ldots\ldots+\beta_{2\rho}-\beta_{2\rho+2}+\beta_{2\rho}-\beta_{2\rho-2}$$
$$+\ldots\ldots+\beta_{2\sigma-1}-\beta_{2\sigma+1}+\beta_{2\sigma-1}-\beta_{2\sigma-3}+\beta_3-\tau_4\beta_1$$
$$+(1-\tau_4)|\beta_1|+(\beta_2-\tau_3\beta_0)+(1-\tau_3)|\beta_0|+|a_1|]$$
$$=|a_n|R^{n+2}+|a_{n-1}|R^{n+1}+R^n[|\alpha_n|+|\beta_n|+|\alpha_{n-1}|+|\beta_{n-1}|$$
$$+2(\alpha_{2\lambda}+\alpha_{2\mu-1}+\beta_{2\rho}+\beta_{2\sigma-1})-(k_1|\alpha_n|+k_3|\beta_n|)$$
$$-(k_2|\alpha_{n-1}|+k_4|\beta_{n-1}|)-(\tau_2\alpha_1+\tau_4\beta_1)-(\tau_1\alpha_0+\tau_3\beta_0)$$
$$+(1-\tau_2)|\alpha_1|+(1-\tau_1)|\alpha_0|+(1-\tau_3)|\beta_1|(1-\tau_4)|\beta_0|+|a_1|]$$
$$=M_1^{*}.$$

For $R\le 1$,

$$|G(z)|\le |a_n|R^{n+2}+|a_{n-1}|R^{n+1}+R[|\alpha_n|+|\beta_n|+|\alpha_{n-1}|+|\beta_{n-1}|$$
$$+2(\alpha_{2\lambda}+\alpha_{2\mu-1}+\beta_{2\rho}+\beta_{2\sigma-1})-(k_1|\alpha_n|+k_3|\beta_n|)$$
$$-(k_2|\alpha_{n-1}|+k_4|\beta_{n-1}|)-(\tau_2\alpha_1+\tau_4\beta_1)-(\tau_1\alpha_0+\tau_3\beta_0)$$
$$+(1-\tau_2)|\alpha_1|+(1-\tau_1)|\alpha_0|+(1-\tau_3)|\beta_1|(1-\tau_4)|\beta_0|+|a_1|]$$
$$=M_1^{**}.$$

Since G(z) is analytic for $|z|\le R$ and G(0)=0, it follows by Schwarz Lemma that $|G(z)|\le M_1^{*}|z|$ for $R\ge 1$ and $|G(z)|\le M_1^{**}|z|$ for $R\le 1$.

Hence, for $R\ge 1$,

$$|F(z)|=|a_0+G(z)|$$
$$\ge |a_0|-|G(z)|$$
$$\ge |a_0|-M^{*}|z|$$
$$>0$$

if $|z|<\dfrac{|a_0|}{M_1^{*}}$.

And for $R\le 1$,

$$|F(z)|=|a_0+G(z)|$$
$$\ge |a_0|-|G(z)|$$
$$\ge |a_0|-M^{**}|z|$$
$$>0$$

if $|z|<\dfrac{|a_0|}{M_1^{**}}$.

This shows that F(z) has no zero in $|z|<\dfrac{|a_0|}{M_1^{*}}$, for $R\ge 1$ and no zero in $|z|<\dfrac{|a_0|}{M_1^{**}}$

for $R \le 1$.

Since the zeros of P(z) are also the zeros of F(z) , it follows that that P(z) has no zero in $|z| < \frac{|a_0|}{M_1^*}$, for $R \ge 1$ and no zero in $|z| < \frac{|a_0|}{M_1^{**}}$ for $R \le 1$, thereby proving Theorem 1 for even n.

The proof for odd n is similar and is omitted.

Proof of Theorem 3: Suppose that n is even and the coefficient conditions hold i.e.

$$k_1|a_n| \le ...... \le |a_{2\lambda+2}| \le |a_{2\lambda}| \ge ...... \ge |a_2| \ge \tau_1 |a_0|$$

$$k_2|a_{n-1}| \le ...... \le |a_{2\mu+1}| \le |a_{2\mu-1}| \ge ...... \ge |a_3| \ge \tau_2 |a_1|.$$

Consider the polynomial

$$\begin{aligned}
F(z) &= (1-z^2)P(z) = (1-z^2)(a_n z^n + a_{n-1}z^{n-1} + ... + a_1 z + a_0) \\
&= -az^{n+2} - a_{n-1}z^{n+1} + (a_n - a_{n-2})z^n + (a_{n-1} - a_{n-3})z^{n-1} + ... + (a_3 - a_1)z^3 \\
&\quad + (a_2 - a_0)z^2 + a_1 z + a_0 \\
&= -a_n z^{n+2} - a_{n-1}z^{n+1} - (k_1 - 1)a_n z^n + (k_1 a_n - a_{n-2})z^n - (k_2 - 1)a_n z^{n-1} \\
&\quad + (k_2 a_{n-1} - a_{n-3})z^{n-1} + (a_{n-2} - a_{n-4})z^{n-2} + (a_{n-3} - a_{n-5})z^{n-3} + \ldots\ldots \\
&\quad + (a_{2\lambda+2} - a_{2\lambda})z^{2\lambda+2} + (a_{2\lambda} - a_{2\lambda-2})z^{2\lambda} + \ldots\ldots + (a_{2\mu+1} - a_{2\mu-1})z^{2\mu+1} \\
&\quad + (a_{2\mu-1} - a_{2\mu-3})z^{2\mu-1} + \ldots\ldots + (a_3 - \tau_2 a_1)z^3 + (\tau_2 a_1 - a_1)z^3 \\
&\quad + (a_2 - \tau_1 a_0)z^2 + (\tau_1 a_0 - a_0)z^2 + a_1 z + a_0 \\
&= a_0 + G(z) \text{, where}
\end{aligned}$$

$$\begin{aligned}
G(z) &= -a_n z^{n+2} - a_{n-1}z^{n+1} - (k_1 - 1)a_n z^n + (k_1 a_n - a_{n-2})z^n - (k_2 - 1)a_n z^{n-1} \\
&\quad + (k_2 a_{n-1} - a_{n-3})z^{n-1} + (a_{n-2} - a_{n-4})z^{n-2} + (a_{n-3} - a_{n-5})z^{n-3} + \ldots\ldots \\
&\quad + (a_{2\lambda+2} - a_{2\lambda})z^{2\lambda+2} + (a_{2\lambda} - a_{2\lambda-2})z^{2\lambda} + \ldots\ldots + (a_{2\mu+1} - a_{2\mu-1})z^{2\mu+1} \\
&\quad + (a_{2\mu-1} - a_{2\mu-3})z^{2\mu-1} + \ldots\ldots + (a_3 - \tau_2 a_1)z^3 + (\tau_2 a_1 - a_1)z^3 \\
&\quad + (a_2 - \tau_1 a_0)z^2 + (\tau_1 a_0 - a_0)z^2 + a_1 z
\end{aligned}$$

For $|z| \le R$, we have by using the hypothesis and Lemma 3

$$\begin{aligned}
|G(z)| &\le |a_n|R^{n+2} + |a_{n-1}|R^{n+1} + (1-k_1)|a_n|R^n + |a_{n-2} - k_1 a_n|R^n + (1-k_2)|a_{n-1}|R^{n-1} \\
&\quad + |a_{n-3} - k_2 a_{n-1}|R^{n-1} + |a_{n-4} - a_{n-2}|R^{n-2} + |a_{n-5} - a_{n-3}|R^{n-3} + \ldots\ldots \\
&\quad + |a_{2\lambda} - a_{2\lambda+2}|R^{2\lambda+2} + |a_{2\lambda} - a_{2\lambda-2}|R^{2\lambda} \ldots\ldots + |a_{2\mu-1} - a_{2\mu+1}|R^{2\mu+1} \\
&\quad + |a_{2\mu-1} - a_{2\mu-3}|R^{2\mu-1} + \ldots\ldots + |a_3 - \tau_2 a_1|R^3 + (1-\tau_2)|a_1|R^3 \\
&\quad + |a_2 - \tau_1 a_0|R^2 + (1-\tau_1)|\alpha_0|R^2 + |a_1|R \\
&\le |a_n|R^{n+2} + |a_{n-1}|R^{n+1} + R^n[(1-k_1)|a_n| + (1-k_2)|a_{n-1}|
\end{aligned}$$

$$+(|a_{n-2}|-k_1|a_n|)\cos\alpha+(|a_{n-2}|+k_1|a_n|)\sin\alpha+(|a_{n-4}|-|a_{n-2}|)\cos\alpha$$
$$+(|a_{n-4}|+|a_{n-2}|)\sin\alpha+......+(|a_{2\lambda}|-|a_{2\lambda+2}|)\cos\alpha+(|a_{2\lambda}|+|a_{2\lambda+2}|)\sin\alpha$$
$$+(|a_{2\lambda}|-|a_{2\lambda-2}|)\cos\alpha+(|a_{2\lambda}|+|a_{2\lambda-2}|)\sin\alpha+......+(|a_{2\mu-1}|-|a_{2\mu+1}|)\cos\alpha$$
$$+(|a_{2\mu-1}|+|a_{2\mu+1}|)\sin\alpha+(|a_{2\mu-1}|-|a_{2\mu-3}|)\cos\alpha+(|a_{2\mu-1}|+|a_{2\mu-3}|)\sin\alpha$$
$$+......+(|a_3|-\tau_2|a_1|)\cos\alpha+(|a_3|+\tau_2|a_1|)\sin\alpha+(1-\tau_2)|a_1|+(1-\tau_1)|a_0|$$
$$+(|a_2|-\tau_1|a_0|)\cos\alpha+(|a_2|+\tau_1|a_0|)\sin\alpha+|a_1|]$$
$$=|a_n|R^{n+2}+|a_{n-1}|R^{n+1}+R^n[|a_1|+(1-k_1)|a_n|+(1-k_2)|a_{n-1}|$$
$$+\cos\alpha(2|a_{2\lambda}|+2|a_{2\mu}|-k_1|a_n|-k_2|a_{n-1}|-\tau_2|a_1|-\tau_1|a_0|)$$
$$+\sin\alpha(k_1|a_n|+k_2|a_{n-1}|+\tau_2|a_1|+\tau_1|a_0|+2\sum_{j=2}^{n-2}|a_j|)]$$
$$=M_5{}^{*}\quad \text{for}\quad R\geq 1.$$

For $R\leq 1$,

$$|G(z)|\leq|a_n|R^{n+2}+|a_{n-1}|R^{n+1}+R[|a_1|+(1-k_1)|a_n|+(1-k_2)|a_{n-1}|$$
$$+\cos\alpha(2|a_{2\lambda}|+2|a_{2\mu}|-k_1|a_n|-k_2|a_{n-1}|-\tau_2|a_1|-\tau_1|a_0|)$$
$$+\sin\alpha(k_1|a_n|+k_2|a_{n-1}|+\tau_2|a_1|+\tau_1|a_0|+2\sum_{j=2}^{n-2}|a_j|)]$$
$$=M_5{}^{**}\ .$$

Since G(z) is analytic for $|z|\leq R$ and G(0)=0, it follows by Schwarz Lemma that $|G(z)|\leq M_5{}^{*}|z|$ for $R\geq 1$ and $|G(z)|\leq M_5{}^{**}|z|$ for $R\leq 1$.

Hence, for $R\geq 1$,

$$\begin{aligned}|F(z)|&=|a_0+G(z)|\\&\geq|a_0|-|G(z)|\\&\geq|a_0|-M_5{}^{*}|z|\\&>0\end{aligned}$$

if $|z|<\dfrac{|a_0|}{M_5{}^{*}}$.

And for $R\leq 1$,

$$\begin{aligned}|F(z)|&=|a_0+G(z)|\\&\geq|a_0|-|G(z)|\\&\geq|a_0|-M_5{}^{**}|z|\\&>0\end{aligned}$$

if $|z|<\dfrac{|a_0|}{M_5{}^{**}}$.

This shows that F(z) has no zero in $|z|<\dfrac{|a_0|}{M_5{}^{*}}$, for $R\geq 1$ and no zero in $|z|<\dfrac{|a_0|}{M_5{}^{**}}$ for $R\leq 1$.

Since the zeros of P(z) are also the zeros of F(z) , it follows that that P(z) has no zero in $|z| < \frac{|a_0|}{M_5^{*}}$, for $R \geq 1$ and no zero in $|z| < \frac{|a_0|}{M_5^{**}}$ for $R \leq 1$, thereby proving Theorem 3 for even n.

For odd n the proof is similar and is omitted.

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