International Journal of Engineering Research and Development
e-ISSN: 2278-067X, p-ISSN: 2278-800X, www.ijerd.com
Volume 9, Issue 2 (November 2013), PP. 57-62

A Mathematical Model on the Two –Phase Hepatic Blood Flow in Arteriols with Special Refrence to Hepatitis B.

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Abstract:- In the present paper we have formulated the Hepatic blood flow in Arteriols. Keeping in view the nature of Hepatic circulatory system in human body. The viscosity increases in the arteoles due to formation of Roulex along axis of red blood cells as we know that the arterioles are remote from heart and proximate to the Liver. **P.N. Pandey** and **V. Upadhyay** have considered that the blood flow as two phased, one of which that of red blood cells and other is plasma. They have also applied the **Herschel bulkley** Non-Newtonian modal in Bio fluid mechanical setup. We have collected a clinical data in case of Hepatitis B for Hematocrit versus blood pressure. The graphical presentation for a particular paramatic value is very close to the clinical observation. The overall presentation is in tensorial form and the solution technic adopted are in analytical as well as numerical. The role of hematocrit is explicit in the determination of blood pressure in case of Hepatic disease Hapatitus B.

I. INTRODUCTION (DESCRIPTION OF BIO- PHYSICAL PROBLEM)

Two phase Hepatic blood flow is a study of measuring the blood pressure if hemoglobin known. The percentage of volume covered by blood cells in the whole blood is called hematocrit. Hematocrit is three times of hemoglobin concentration (reported as grams per deciliter)**.[1].**

This work will focus on two phase hepatic blood flow in arterioles with special reference to HepatitisB. There are many work is available in that field, but P.N.Pandey and V. Upadhyay (2001) discussed a some phenomena in two phase blood flow gave an idea on the two phase hepatic blood flow in arterioles with a Liver disease HepatitisB. The work of P.N. Pandey and V.Upadhyay in whole circulatory system but this work will focus on Hepatitc circulatory system, and Hepatitic circulatory system is a sub system of whole circulatory system. In this work, applied the Herschel Bulkley non-Newtonian model.

Blood is a complex fluid consisting of particulare corpulse suspended in a non-Newtonian fluid. The particulare solids are red blood cells (RBCs), white blood cells (WBCs) and platelets. The fluid is plasma, which itself is a complex mixture of proteins and other intergradient in an aqueous base. In blood 98% RBC and remaining 2% WBC and platelets cells i.e. there are a few part of the other cells. Which are ignorable, so one phase of the blood is plasma and 2^{nd} phase of the blood is RBCs.

Liver is the largest gland in body. It is a reddish brown in the body with four lobes of un equal size and shape. These are right lobe, left lobe, quadrats and caudate lobe.A human Liver is normaly weight 1.44-1.66 kg. Liver lobes are sarrounded by a thick capsule, mostly overlaid with reflected periforinum [**2**].

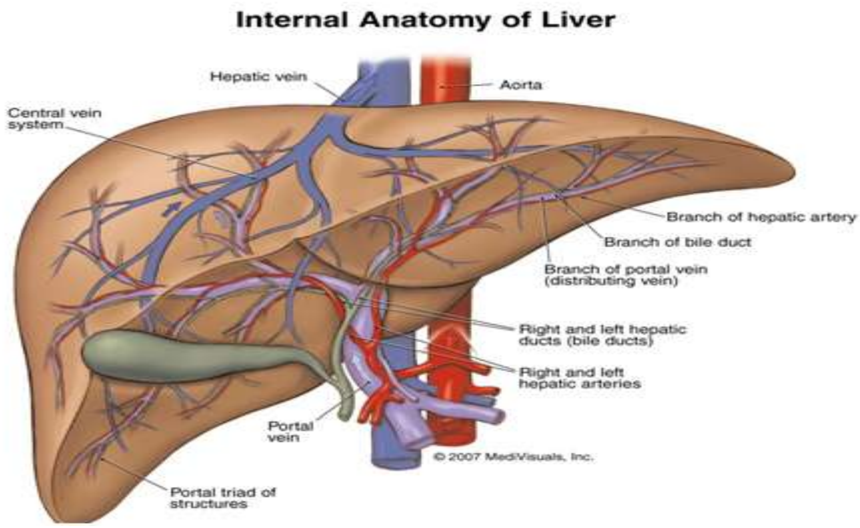


Fig 1

The Liver recive a dual blood supply from the hepatic portal vein and hepatic arteries supplying approximately 75% OF Liver blood supply. Liver volume and portal blood flow decreases after the age of 50 [**3**].

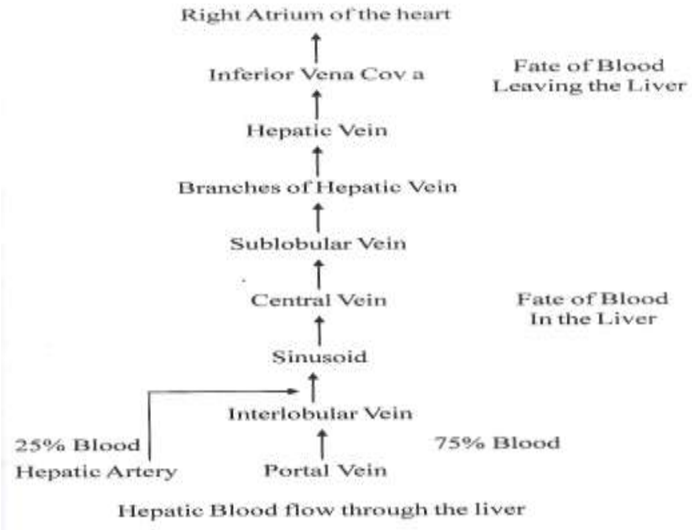


fig :-2

Liver is also prone to many diseases [**4**]. One of these is Hepatitis B. Hepatitis is irritation and swelling (Inflamation) of liver due to infection with the Hepatitis B virus(HBV). In 1947 Mac Callum classified viral Hepatitis A & viral hepatitis B . Hepatitis B virus was discovered in 1965 by Dr. Baruch Blumberg. In 1981 The FDA approved sophisticated Plasma, Derived Hepatitis vaccine are human use . In 2001,FDA approved a combination Hepatitis A and Hepatitis B vaccine (Twinrix, Glaxo Smithkline). During 1990-2004, incidence of acute Hepatits B in the United States declined 75%. The greatest decline (94%) occurred among children and adolescent. A total of 3405 cases of Hepatitis B were reported in 2009.In 2010, India has 40 million and China has 120 million people are infected with Hepatitis B. WHO , estimated 600,000 people die every year related to the infection. In India alone, the death rates from Hepatitis B exceed over 100,000 per year.

II. MATHEMATICAL MODELING

Basic Bio-fluid equation for two phase blood flow-

Let us the problem of blood flow in hepatic circulatory system is different from the problems in cylindrical tube and select generalized three dimensional orthogonal curvilinear coordinate system. Briefly described as E^3 called as Euclidean space. According to mishra the biophysical laws thus expressed fully hold good in any co-ordinate system which is a compulsion for the truthfulness of the laws (1990).

According to Sherman I.W. and Sherman V.,G Blood is mixed fluid. Mainly there are two phases in blood. The first phase is plasma, while the other phase is that of blood cells are enclosed with a semi-permeable membrance whose density is greater than that of plasma. These blood cells are uniformly distributed in plasma. Thus, blood can be considered as a homogeneous mixture of two phases (1989).

Equation of Continuity for two phase blood flow-

According to Singh P. and Upadhyay K.S. The flow of blood is affected by the presence of blood cells. This effect is directly proportional to the volume occupied by blood cells. Let the volume portion covered by blood cells in unit volume be X , this X is replaced by H/100, where H is the Hematocrit the volume percentage of blood cells .Then the volume portion covered by the plasma will be 1-X . If the mass ratio of blood cells to plasma is r then clearly

$$r = \frac{X\rho_c}{(1-X)\rho_p} \qquad (1)$$

Where ρ_c and ρ_p are densities of blood cells and blood blood plasma respectively. Usually this mass ratio is not a constant, even then this may be supposed to constant in present context (1986)

The both phase of blood, i.e. blood cells and plasma move with the common velocity .Campbell and Pitcher has presented a model for this situation .According to this model, we consider the two phase of blood separately (1958) .Hence equation of continuity for two phase according to the principle of conservation of mass defined by J.N and Gupta R.C. as follow

$$\frac{\partial(X\rho_c)}{\partial t} + (X\rho_c V^i)_{,i} = 0 \qquad (2)$$

and

$\frac{\partial(1-X)\rho_p}{\partial t}$ + ((1-X)$\rho_{p\,V^i}$),i =0 (3)

Where v is the common velocity of two phase blood cells and plasma .Again (X$\rho_{c\,V^i}$),i is co-variant derivative of (X$\rho_{c\,V^i}$) with respect to X^i . In the same way (1-X)ρ_c V^i),i is co-variant derivative of ((1-X)ρ_c V^i) w.r. to X^i .

If we define the uniform density of the blood ρ_m as follow

$\frac{1+r}{\rho_m} = \frac{r}{\rho_c} + \frac{1}{\rho_p}$ (4)

Then equation (2) and (3) can be combined together as follow [**10**]

$\frac{\partial\rho_m}{\partial t}$ +($\rho_{m\,V^i}$),i = 0 (5)

Equation of Motion for two phase blood flow-

According to Ruch, T.C. and H.D. The hydro dynamical pressure p between the two phases of blood can be supposed to be uniform because the both phases i.e. blood cells and plasma are always in equilibrium state in blood (1973). Taking viscosity coefficient of blood cells to be η_c and applying the principle of conservation of momentum, we get the equation of motion for two phase of blood cells as follows:

$X\rho_c \frac{\partial V^i}{\partial t} + (X\rho_c V^j)V^i_{,j} = -Xp_{,j}g^{ij} + X\eta c\left(g^{jk}V^i_{,k}\right)_{,j}$ (6)

Similarly, taking the viscosity coefficient of plasma to be. The equation of motion for plasma will be as follows:

$(1-X)\rho_p \frac{\partial V^i}{\partial t} + \{(1-X)\rho_p V^i\}V^i_{,j} = -(1-X)p_{,j}g^{ij} + (1-X)\eta c\left(g^{jk}V^i_{,k}\right)_{,j}$ (7)

Now adding equation (6) and (7) and using relation (4), the equation of motion for blood flow with the both phases will be as follows:

$\rho_m \frac{\partial V^i}{\partial t} + (\rho_m V^j)V^i_{,j} = -p_{,j} + \eta_m\left(g^{jk}V^i_{,k}\right)_{,j}$ (8)

Where η_m =Xη_c + $(1-X)\eta_p$ is the viscosity coefficient of blood as a mixture of two phases.

As the velocity of Blood flow decreases, the viscosity of blood increases. The velocity of blood deceases successively because of the fact that arterioles, veinules and veins these vessels are relatively a far enough from the heart. Hence the pumping of the heart on these vessels is relatively low [**8**]. Secondly these vessels relatively narrow down more rapidly. In this situation, the blood cells line up on the axis to build up rouleaux.

Hence a yield stress is produced. Though this yield stress is very small, even then the viscosity of blood is increased nearly ten times [**9**].

The Herschel Bulkley law holds good on the two phase blood flow through veins arterioles, veinules and whose constitutive equation is as follows:

T'=η_m e^n+ T_p (T'≥ T_p) and

e =0 (T'< T_p) where,

T_p is the yield stress.

When strain rate e = 0 (T'< T_p) a core region is formed which flows just like a plug. Let the radius of the plug be r_p. The stress acting on the surface of plug will be T_p. Equating the forces acting on the plug, we get [fig (1)]

$P\pi r_p^2 = T_p 2\pi r_p$

=> $r_p = 2\frac{T_p}{P}$ (9)

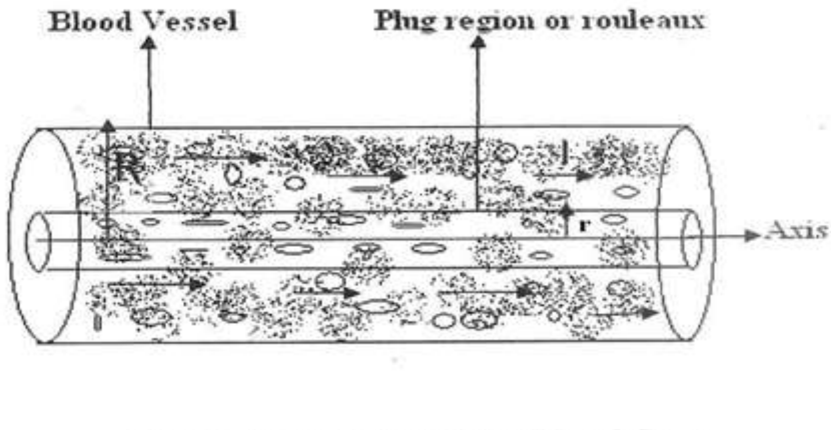


Fig (3): Herschel Bulkley blood flow

The Constitutive equation for test part of the blood vessel is

$T' = \eta_m e^n + T_p$ or $T' - T_p = \eta_m e^n = T_e$

Where, T_e = effective Stress

Whose generalized form will be as follows"

$T^{ij} = -P g^{ij} + T_e^{ij}$ where, $T_e^{ij} = \eta_m (e^{ij})^n$ While $e^{ij} = g^{jk} V_k^i$

Where the symbols have their usual meanings.

Now we describe the basic equations for Herschel Bulkley blood flow as follows:

Equation of Continuity-

$1/\sqrt{g}(\sqrt{g}v^i)_{,i} = 0$

Equation of Motion-

$$\rho_m \frac{\partial v^i}{\partial t} + \rho_m v^i v^i_{,j} = -T_{e,j}^{ij} \qquad (10)$$

Where all the symbols have their usual meanings.

Solution & Discussion

Since, the blood vessels are cylindrical; the above governing equations have to be transformed into cylindrical co-ordinates. As we know earlier:

$X^1 = r, \quad X^2 = \theta, \quad X^3 = Z,$

Matrix of metric tensor in cylindrical co-ordinates is as follows:

$$[g_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

While matrix of conjugate metric tensor is as follows:

$$[g^{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Whereas the chritoffel's symbols of 2nd kind are as follows:

$\left\{ \begin{matrix} 1 \\ 2\ 2 \end{matrix} \right\} = -r, \quad \left\{ \begin{matrix} 1 \\ 2\ 2 \end{matrix} \right\} = \left\{ \begin{matrix} 1 \\ 2\ 2 \end{matrix} \right\} = \frac{1}{r}$ remaining others are zero.

Relation between contravariant and physical components of velocity of blood flow will be as follows:

$\sqrt{g_{11 v^1}} = v_r \Rightarrow v_r = v^1$

$\sqrt{g_{22 v^2}} = v_\Theta \Rightarrow v_\Theta = r\, v^2$

$\sqrt{g_{33 v^3}} = v_z \Rightarrow v_z = v^3$

Again the physical components of $-p_{,j} g^{ij}$ are $-\sqrt{g_{ij}}\, p_{,j} g^{ij}$

Equations (9) and (10) are transformed into cylindrical form so as to solve them as power law model to get

$$\frac{dv}{dr} = \left(\frac{Pr}{2\eta_m}\right)^{\frac{1}{n}}$$

Where, pressure gradient $\frac{dp}{dz} = P$

Replace r to r- r_p, for non – plug region

$$-\frac{dv}{dr} = \left(\frac{P(r-r_p)}{2\eta_m}\right)^{\frac{1}{n}}$$

$$-\frac{dv}{dr} = \left(\frac{\frac{1}{2}Pr - \frac{1}{2}Pr_p)}{\eta_m}\right)^{\frac{1}{n}}$$

From equation (9)

$$-\frac{dv}{dr} = \left(\frac{\frac{1}{2}Pr}{\eta_m}\right)^{\frac{1}{n}} \qquad (11)$$

Substituting the value of T_p from (7) into (11), we get

$$-\frac{dv}{dr} = \left(\frac{\frac{1}{2}Pr - \frac{1}{2}Pr_p}{\eta_m}\right)^{\frac{1}{n}}$$

$$\frac{dv}{dr} = -(P/2\eta_m)^{\frac{1}{n}} (r - r_p)^{\frac{1}{n}} \qquad (12)$$

Integrating above equation (12) under the no slip boundary condition: v=0 at r=R so as to get:

$$v = (P/2\eta_m)^{\frac{1}{n}} \frac{n}{n+1} \left[(R - r_p)^{\frac{1}{n}+1} - (r - r_p)^{\frac{1}{n}+1} \right] \qquad (13)$$

This is the formula for velocity of blood flow in arterioles, veinules and veins.

Putting r=r_p to get the velocity v_p of plug flow as follows:

$$v_p = \frac{n}{n+1}(P/2\eta m)^{\frac{1}{n}}(R-r_p)^{\frac{1}{n}+1} \qquad (14)$$

Where the value of r_p is taken from (7).

Result **(Bio-physical interpretation) -**

Observations: Hematocrit Vs. Blood pressure is taken from Lala Lajpat Rai and Associated Hospital, Kanpur by **Dr. Sanjay Verma.**

Patient Name: - Mr. Vijay Chandra Annual No. 19667/2011

Diagnosis: -

Sr. No.	Date	HB(Hemoglobin)	B.P.(blood Pressure)	Hematocrit
1	04.09.11	5.69 gm/dl	96/60	17.07
2	05.09.11	5.7 gm/dl	100/60	17.10
3	08.09.11	5.0 gm/dl	90/60	15.00
4	09.08.11	7.5 gm/dl	130/80	22.50

The flow flux of two phased blood flow in arterioles, veinules and veins is

$$Q = \int_0^{r_p} 2\pi r\, v_p\, dr + \int_{r_p}^{R} 2\pi r v dr$$

$$= \int_0^{r_p} 2\pi r \frac{n}{n+1}\left(\frac{P}{2\eta m}\right)^{\frac{1}{n}}(R-r_p)^{\frac{1}{n}+1} dr + \int_0^{r_p} 2\pi r \frac{n}{n+1}(P/2\eta m)^{\frac{1}{n}}\left[(R-r_p)^{\frac{1}{n}+1} - (r-r_p)^{\frac{1}{n}+1}\right]dr$$

Using (12) and (14)

$$= \frac{2\pi n}{(n+1)}\left(\frac{P}{2\eta m}\right)^{\frac{1}{n}}(R-r_p)^{\frac{1}{n}+1}\left[\frac{r^2}{2}\right]_0^{r_p} + \frac{2\pi n}{(n+1)}\left(\frac{P}{2\eta m}\right)^{\frac{1}{n}}\left[\frac{r^2}{2}(R-r_p)^{\frac{1}{n}+1} - \frac{r(r-r_p)^{\frac{1}{n}}}{\frac{1}{n}+2} + \frac{(r-r_p)^{\frac{1}{n}+3}}{\left(\frac{1}{n}+2\right)\left(\frac{1}{n}+3\right)}\right]_{r_p}^{R}$$

$$= \frac{\pi n}{(n+1)}\left(\frac{P}{2\eta m}\right)^{\frac{1}{n}} R^{\frac{1}{n}+3}\left[\frac{r_p^2}{R^2}\left(1-\frac{r_p^2}{R}\right)^{\frac{1}{n}+1} + \left(1+\frac{r_p}{R}\right)\left(1-\frac{r_p}{R}\right)^{\frac{1}{n}+2} - \frac{2\left(1-\frac{r_p}{R}\right)^{\frac{1}{n}+2}}{\left(\frac{1}{n}+2\right)} + \frac{2\left(1-\frac{r_p}{R}\right)^{\frac{1}{n}+3}}{\left(\frac{1}{n}+2\right)\left(\frac{1}{n}+3\right)}\right]$$

Q= 1000 ml. /min R =1, $r_p = \frac{1}{3}$

According to Gustafson, Daniel R. (1980)

η_p = 0.0015 (Pascal-sec.)

According to Glenn Elert (2010)

η_m = 0.035 (pascal-sec.)

H = 15.0 , P= 90

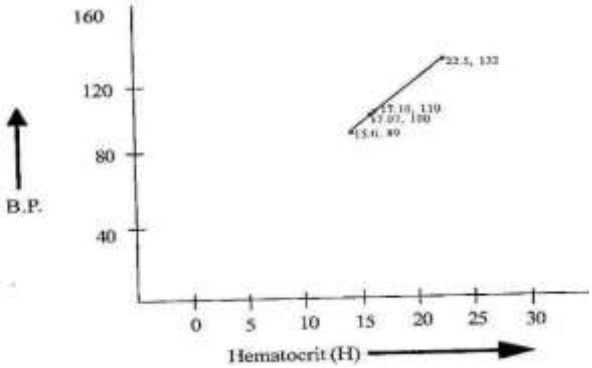
$\eta_m = \eta_c X + \eta_p (1-X)$ where, $X = \frac{H}{100}$

$$4299.36 = (857.1428)^{\frac{1}{n}}\left[\frac{26n^3+33n^2+9n}{6n^3+11n^2+6n+1}\right]$$

Solved by Numerical method,

We get, **n = 0.91843**

From equation (5.1), We have



$$= (0.006699\,H+0.0045)\left(\frac{27Q}{2\pi}\right)^{n} \times \left(\frac{(26n^3+33n^2+9n)}{(6n^3+11n^2+6n+1)}\right)^{n}$$

P = (0.006699 H+0.0045)(857.3567)

P = (5.7434) H + 3.8581

At H = 17.07 $P = 99.94 \approx 100$ At H = 17.10, $P = 101.11 \approx 101$

At H=15.00 $P = 89.17 \approx 89$ At H= 22.5, P=131.84 ≈ 132

Graph: (1)

III. CONCLUSION

A simple survey of the graph between blood pressure and hematocrit in Hepatitis B patient shows that when hematocrit increased then Blood pressure is also increased, That is the hematocrit is proportional to blood pressure.

Acknowledgement:

I give very sincere thanks to **Dr**. **Sanjay Verma** , physician of Lala Lajpat Rai and Associated Hospital, Kanpur.

Remark:

If this would have been possible to get blood Pressure on the particular tissue (Liver) during operation of HepatitisB patient then the relation between blood pressure and hemoglobin has been measured more accurately.

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